The Beta Transmuted Mukherjee-Islam Distribution: Estimation and Applications

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Keywords

1. Mukherjee-Islam distribution
2. Rank Transmutation Map
3. Beta generalized distribution
4. Beta Transmuted Mukherjee-Islam distribution
5. Maximum likelihood estimation
6. Information Criterion

Abstract:

This research proposes a novel five-parameter statistical distribution called the Beta Transmuted Mukherjee-Islam (BTM-I) distribution. This distribution has been developed to enhance the capability and flexibility of statistical modeling for various data characteristics. It integrates the Beta technique and the quadratic rank transmutation map method with the Mukherjee-Islam distribution. The proposed distribution encompasses several competing models as special cases and is characterized by five parameters, making it more flexible in representing diverse data types.

The study derives the mathematical properties of the new distribution, including moments, moment generating function, mean deviation, order statistics, and reliability analysis. The estimation of the parameters is performed using the Maximum Likelihood method (MLE). The accuracy of these estimators is evaluated through an extensive simulation study of key performance metrics like Bias, Standard Error (SE), and Mean Squared Error (MSE). Crucial measures for assessing confidence intervals, such as Coverage Probability (CP) and Coverage Length (CL), were also utilized. The results demonstrated that the maximum likelihood estimators were consistent and the obtained CP and CL values were satisfactory, confirming the reliability and effectiveness of the estimations.

The study also includes a practical application to three real-world datasets to illustrate the significance, efficiency, and utility of the proposed distribution. The performance of the BTM-I distribution was compared to that of several competing statistical models. The results conclusively showed that the BTM-I distribution possesses superior flexibility that enables it to effectively model various types of data and outperform the competing models. These findings confirm that the BTM-I distribution is a robust and effective tool and represents a valuable addition for statisticians and researchers across multiple fields.

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توزيع مواخرجي–اسلام المحول بيتا: التقدير والتطبيقات

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الكلمات المفتاحية

1. توزيع مواخرجي_ اسلام(M-I)
2. خريطة تحويل الرتبة (RTM)
3. توزيع بيتا المعمم(B-G)
4. توزيع مواخرجي_اسلام المحول بيتا (BTM-I)
5. طريقة الإمكان الأعظم (MLE)
6. معايير جودة المطابقة (IC)

الملخص:

قدمت هذه الدراسة توزيعًا إحصائيًا جديدًا ذا خمس معلمات، يُعرف باسم توزيع مواخرجي–اسلام المحول بيتا (BTM–I)، والذي تم تطويره لتعزيز قدرة ومرونة النمذجة الإحصائية لخصائص البيانات المتنوعة. يدمج هذا التوزيع تقنية بيتا مع طريقة خريطة التحويل الرتبي التربيعية التي اقترحها شو وزملاؤه (2007)، وذلك لتوسيع نطاق توزيع مواخرجي–اسلام الأصلي. التوزيع المقترح يشمل عدة نماذج منافسة كحالات خاصة، وتمنحه معلماته الخمس مرونة أكبر في تمثيل أنواع مختلفة من البيانات. تناولت الدراسة الخصائص الرياضية الأساسية للتوزيع الجديد، بما في ذلك العزوم، دالة توليد العزوم، متوسط الانحراف، إحصائيات الرتبة، وتحليل الموثوقية. لتقييم الأداء العملي لتوزيع (BTM–I) ، تم تقدير معلماته باستخدام طريقة الامكان الأعظم (MLE)، وتم تقييم دقة هذه المقدّرات من خلال دراسة محاكاة شاملة شملت مقاييس أداء رئيسية مثل التحيز (Bias) ، الخطأ المعياري (SE) ، ومتوسط مربع الخطأ. (MSE)

كما تم استخدام مقاييس حاسمة لتقييم فترات الثقة، وهي احتمالية التغطية (CP) وطول التغطية (CL)، وقد أظهرت النتائج أن مقدّرات الامكان الأعظم كانت متسقة، وأن قيم CP وCL كانت مرضية، مما يؤكد موثوقية وفعالية التقديرات. تم تطبيق التوزيع المقترح على ثلاث مجموعات بيانات واقعية لإثبات أهميته وفعاليته، حيث أظهرت النتائج بشكل قاطع أن توزيع (BTM–I) يمتلك مرونة فائقة، مما يمكّنه من نمذجة أنواع مختلفة من البيانات بفعالية ويتفوق على النماذج المنافسة. تؤكد هذه النتائج أن توزيع (BTM–I) هو أداة قوية وفعالة، ويمثل إضافة قيّمة للإحصائيين والباحثين في مجالات متعددة.

1. Introduction

communications, life-testing and many others. The well-known and fundamental distributions, such as exponential, Rayleigh, Weibull and gamma, are still quite limited in terms of some characteristics and capability of showing wide flexibility. For example, the exponential distribution is capable of modeling with constant hazard function, whereas the Rayleigh distribution has increasing hazard function only. However, the Weibull is much

2. To derive a number of possible statistical functions for the new (BTM-I) distribution such as the cumulative distribution function, probability density function, survival function, hazard rate function, odds function, quantiles function, as well as deriving a number of possible statistical properties of the distribution, such as the moments, the function generating the moments, the mean and median deviations, entropy, order statistics.
3. To estimate the unknown distribution parameters using the maximum likelihood method.
4. To study the performance of the Maximum Likelihood Estimators for the parameters of the (BTM-I) distribution using simulation by SAS program.
5. To investigate the confidence intervals for the parameters of the new distribution.
6. The importance and flexibility of the proposed distribution are illustrated by application to two real data sets.

Significance of the Study

Probability distributions are considered the base of all statistical methods and applicative.

Since there is a clear need for extended forms of these distributions, a significant progress has been made towards the generalization of some

Researchers resort to generalizing the most efficient, flexible, and appropriate distributions for the data in order to obtain the lowest estimated value for the parameters of the new distribution among other basic distributions.

flexible and capable of modeling with increasing, decreasing or constant hazard function.

This indicates that there are instances where classical distributions do not provide adequate fits to real data. It is due to this shortcoming that several probability distributions have been developed by introducing one or more parameters to generate new distributions. In literature, new families of distributions have been developed by introducing shape parameters to control skewness, kurtosis and tail weights, thus providing great flexibility in modelling skewed data. In this study, the genesis of the Beta distribution and quadratic rank transformation map (QRTM) presented by Buckley et al. (2009) is used to develop the so-called transmuted Mukherjee-Islam distribution and obtain a new distribution called Beta Transmuted Mukherjee-Islam (BTM-I). Specialized statistical programs have contributed to the calculation of mathematical and statistical properties, and this has led to an increase in these generalizations, which in turn has increased the number of statistics researchers attempting to develop new distributions.

Statistical distributions play a vital role when it comes to description and prediction of real data in various research areas such as engineering, financial modelling, economics, biological and reliability studies. The recent development in distribution theory focuses on problem-solving issues facing researchers and proposes a variety of models that can better investigate and assess lifetime data sets across different applied areas. In other words, there is a need to introduce more efficient models capable of exploring the real phenomenon of nature. Nowadays, the trends and practices in proposing new probability models totally differ in comparison to the models suggested prior to 1997. One main objective of proposing, extending or generalizing (models or their classes) is to explain how the lifetime phenomenon arises in fields like physics, computer science, insurance, public health, medical, engineering, biology, industry, The Beta-G class of distributions generalizes a base distribution G, whose cumulative distribution function cdf is given by $G(x)$. Specifically, when the parameters a and b are both equal to 1, the cdf of the Beta-G distribution simplifies to $G(x)$, effectively coinciding with the primitive distribution. The inclusion of the additional parameters a and b allows for the introduction of skewness and provides control over the tail weights of the distribution. This added flexibility in well-known distributions and their successful application to problems in areas such as engineering, finance, economics and biomedical sciences, among others. Consequently, the development of new generalizations to improve the goodness-of fit of classic distributions is necessary to obtain estimators for the parameters that have less values compared to sub-models. The importance of this study lies in developing a new generalization of the transmuted Mukherjee-Islam distribution, which will be called the Beta Transmuted Mukherjee-Islam and applied to a set of real data.

Key terms and definitionsBeta-Generated

Eugene et al. (2002) first introduced the class of generalized Beta distributions through its cumulative distribution function (cdf). This cdf serves as the foundation for defining this versatile family of probability distributions. The cdf of the generalized Beta distribution is expressed as:

where $G(x)$ is the cdf of a parent random variable and the Beta function is:

$$\boldsymbol{F(x)=\frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}(1-t)^{b-1}\,dt, a>0, b>0} \quad (1)$$

Where $|\lambda|\le 1$ is the transmuted parameter and $F(x)$ is the cdf of the base distribution. This added flexibility allows for adjustments in the shape and characteristics of the resulting distribution. The parameter λ influences the extent of the transformation: when λ = 0, the new distribution is identical to the base distribution; when λ > 0, the new distribution tends to be more right-skewed; and when λ < 0, it leans towards left-

Statement of the Problem

Probability Practical applications in various fields of applied sciences have shown that many of the continuous basic distributions do not have enough flexibility. To get more flexible distributions, many researchers have proposed new generalizations for the classic distributions via quadratic rank transmutation map. In this study the genesis of the Beta distribution and transmuted map is used to develop the so-called Beta transmuted Mukherjee-Islam distribution through inserting additional shape parameters to propose a new distribution that is expected to be highly flexible and efficient.

Objectives of the Study

This study aims to achieve the following objectives:

1. To develop a new (BTM-I) distribution by adding two extra shape parameters for the cumulative distribution function in transmuted Mukherjee-Islam to provide more flexibility to the generated family.

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\,dx.$$

The Beta-G class of distributions generalizes a base distribution G, whose cdf is given by $G(x)$. Specifically, when the parameters a and b are both equal to 1, the cdf of the Beta-G distribution simplifies to $G(x)$, effectively coinciding with the primitive distribution. The inclusion of the additional parameters a and b allows for the introduction of skewness and provides control over the tail weights of the distribution. This added flexibility in shaping the distribution makes the Beta- G family a powerful tool for modeling observed data. The original distribution with cdf $G(x)$ is often referred to as the primitive distribution. The cdf of Beta-G can be rewritten as:

$$F(x)=I_{G(x)}(a,b)=\frac{B_{G(x)}(a,b)}{B(a,b)},$$

where $B_{G(x)}(a,b)$ denotes the incomplete Beta function.

$$G(x) = F(x)(1+\lambda) - \lambda F(x,\varphi)^2.$$

$$g(x) = f(x)(1+\lambda-2\lambda F(x)).$$

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$$g(x) = f(x)(1+\lambda-2\lambda F(x)).$$

Where $|\lambda| \le 1$ is the transmuted parameter and $F(x)$is the cdf of the base distribution. A random variable $X$is said to have transmuted Mukherjee-Islam distribution with parameters $p > 0, \theta > 0$ and $|\lambda| \le 1$,if it has the cdf is

$$G(x) = \left(\frac{x}{\theta}\right)^p \left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right), \quad (5)$$

the corresponding pdf of the transmuted Mukherjee-Islam distribution is given by

$$g(x) = \frac{p}{\theta^p}x^{p-1}\left(1+\lambda-2\lambda\left(\frac{x}{\theta}\right)^p\right). \quad (6)$$

3 Beta Transmuted Mukherjee-Islam Distribution

Generalizing distributions have been frequently discussed in statistics in terms of problems of trying to fit and model observed data in various areas. A generalized class of the Beta distribution was first given by Eugene, et. al (2002), where the Beta Normal distribution was introduced as a generalization of the Normal distribution. A class of generalized distributions $F(x)$ has received considerable attention over the last few years, particularly after the studies of Eugene, et. al (2002) and Jones (2004). If $F(x)$ denotes the baseline cdf of a random variable, then the Beta generalized distribution is defined as

$$G(x) = \left(\frac{x}{\theta}\right)^p, \quad (3)$$

$$g(x) = \frac{p}{\theta^p}x^{p-1}. \quad (4)$$

which is one of its most important properties in reliability analysis. Its mathematical form is simple and can be handled easily, and this is the reason for preferring it over other more complex distributions such as normal, weibull, Beta etc. A random variable X is said to have a Mukherjee skewed. The QRTM has been successfully applied to various base distributions, such as the normal, exponential, and Weibull, leading to the development of new distribution families with potential advantages in modeling real-world phenomena.

Preliminaries

1 Mukherjee-Islam Distribution

Mukherjee-Islam distribution was introduced by Mukherjee and Islam (1983). Many researchers have considered the Mukherjee-Islam model as a lifetime distribution and applied it in various areas of statistical applications. Siddiqui (1992), for example, studied the physical properties of the Mukherjee-Islam distribution and discovered that the distribution could be monotonic decreasing and exhibit unimodal failure rates. The modification of Mukherjee-Islam distribution was discussed in Siddiqui et al (1997) where Lawless's approach was applied (see Lawless, 2011). It is finite range distribution, shaping the distribution makes the Beta- G family a powerful tool for modeling observed data. The original distribution with cdf $G(x)$ is often referred to as the primitive distribution. The cdf of Beta-G can be rewritten as:

$$F(x) = I_{G(x)}(a,b) = \frac{B_{G(x)}(a,b)}{B(a,b)},$$

where $B_{G(x)}(a,b)$ denotes the incomplete Beta function given by:

$$B_{G(x)}(a,b) = \int_0^{G(x)} t^{a-1}(1-t)^{b-1}\,dt, a>0, b>0, \quad (2)$$

and $I_{G(x)}(a,b)$ is the incomplete Beta ratio function.

Quadratic Rank Transmutation Map

The Quadratic Rank Transmutation Map (QRTM), which was proposed by Shaw and Buckley in 2007, is a method for creating new probability distributions from existing ones.

This technique involves a simple transformation that adds a quadratic term, controlled by a parameter (λ), to the base distribution's cdf; the cdf and pdf of the T-G family of distributions are defined, respectively, by:

$$F(x) = \sum_{j=1}^{\infty}(-1)^j\binom{b-1}{j}\frac{\left(\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right)^{a+j}}{B(a,b)(a+j)}. \quad (13)$$

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}(1-t)^{b-1}\,dt, \quad (7)$$

$$F(x) = \frac{1}{B(a,b)}\int_0^{\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)} t^{a-1}(1-t)^{b-1}\,dt, \quad (8)$$

where $x>0, p>0, \theta>0, |\lambda|\le 1, a>0,$

and $b>0$. The cdf can be expressed in a closed form using the hypergeometric function (see Cordeiro and Nadarajah 2011) as follows:

$$F(x) = \frac{\left(\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right)^a}{a\,B(a,b)}.2F1\left[a,1-b,a+1;\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right] \quad (9)$$

Where $(\theta>0)$ is the scale parameter and its $p,a,b,\lambda>0$ are shape parameters. Note that $I_y(a,b)=\frac{B_y(a,b)}{B(a,b)}$ is the incomplete Beta ratio function, and $B_y(a,b)=\int_0^y t^{a-1}(1-t)^{b-1}\,dt$ is the incomplete Beta function, $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function and $\Gamma(.)$ is the gamma function.

Mathematical and Statistical Properties

1. Survival Function

The probability density function (pdf) of the Beta transmuted Mukherjee-Islam distribution has the form:

$$f(x) = \frac{g(x)}{B(a,b)}[G(x)]^{a-1}[1-G(x)]^{b-1}. \quad (10)$$

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \frac{\left(\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right)^a}{a\,B(a,b)} * 2F1\left[a,1-b,a+1;\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right]. \quad (15)$$

Islam distribution if its cdf and the probability density

Where $p,\theta>0$ *and* $\theta > x > 0$. Here p *and* θ *are* shape and scale parameters.

2 Transmuted Mukherjee-Islam Distribution

Recently, many authors used the Transmuted Generalization (T-G) family to propose new generalizations of some well-known distributions. For example, Aryal and Tsokos (2009) defined the transmuted generalized extreme value. For a detailed study on the general properties of the transmuted family of distributions, one can see Bourguignon, Ghosh and Cordeiro (2015). Consider a baseline cdf and pdf. Then, the cdf and pdf of the T-G family of distributions are respectively, defined by:

By compensating for $g(x)$ and $G(x)$ in (10) we get the pdf of the BTM-I distribution given by

$$f(x) = \frac{1}{B(a,b)}\frac{p}{\theta^p}x^{p-1}\left(1+\lambda-2\lambda\left(\frac{x}{\theta}\right)^p\right)\left[\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right]^{a-1}\left[1-\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right]^{b-1}, \quad (11)$$

where $x>0$, $\theta>0$ is the scale parameter, and the shape parameter is $p>0$.

Mixture representation

In this section we find the series representations of the cdf and pdf of the BTM-I distribution, which will be useful for studying its mathematical characteristics. As we shall see, both the pdf and cdf of BTM-I distribution can be expressed in terms of the Mukherjee-Islam distribution. By using (7) and the power series expansion of $(1-t)^{b-1}$, we get:

$$F(x) = \frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}(1-t)^{b-1}\,dt = \sum_{j=1}^{\infty}(-1)^j\binom{b-1}{j}\frac{G(x)^{a+j}}{a+j}. \quad (12)$$

with the binomial $\binom{b-1}{j}=\frac{\Gamma(b)}{\Gamma(b-j)\,j!}$ term defined for any real. Hence, (12) reduces

$$w_{kl} = \sum_{j=0}^{\infty} (-1)^{j+k+1} \binom{b-1}{j} \binom{a+j}{k} \binom{a+j}{l} \frac{\lambda^l}{B(a,b)(a+j)},$$

and $g(x;\theta,p(k+l))$ is the Mukherjee-Islam pdf with scale θ and shape $p(k+1)$ parameters. If $b>0$ is an integer, the index $b>0$ in the sum stops at $b-1$, and if a is an integer, then the indices k and l in the sum stop at $a+j$. Thus, several mathematical properties of the BTM-I distribution can be obtained simply from those properties of the exp-G family. Equations (13) and (14) are the main result of this section.

1. Survival function

Survival function $S(x)$ is important in calculating system's survival probability. Let X be a continuous random variable having cdf $F(x)$ and pdf $f(x)$, mathematically

5. Moments and Moment Generating Function

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g. mean, dispersion, skewness and kurtosis). Using the equation (2.34), the r^{th}moment of the BTM-I random variable is given by:

$$E(x^r) = \int_{-\infty}^{\infty} x^r f(x) dx$$

$$E(x^r) = \int_{-\infty}^{\infty} x^r \sum_{k,l=0}^{\infty} w_{kl} f(x;\theta,p(k+l)) dx$$

$$= \int_{-\infty}^{\infty} x^r \sum_{k,l=0}^{\infty} w_{kl} \frac{p}{\theta}(k+l)\left(\frac{x}{\theta}\right)^{p(k+l)} dx$$

$$= \sum_{k,l=0}^{\infty} w_{kl} \sum_{r=0}^{\infty} \theta^r \frac{(k+l)p}{p(k+l)+r}, \qquad (16)$$

The moment generation function (MGF) of a random variable X is defined by:

2. Hazard Function

The mathematical formula for hazard function which is otherwise called failure rate is: $h(x) = \frac{f(x)}{S(x)}$; so, we obtain the hazard function of the BTM-I distribution as:

$$h(x) = \frac{\sum_{k,l=0}^{\infty} w_{kl} \frac{p(k+l)}{\theta^{p(k+l)}} x^{p(k+l)-1}}{1 - \frac{\left(\left(\frac{x}{\theta}\right)^p \left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right)^a}{a\,B(a,b)} . 2F1\left[a, 1-b, a+1; \left(\frac{x}{\theta}\right)^p \left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right]}$$

where $0 < x < \theta,\ p > 0,\ b > 0,\ a > 0, \theta > 0$

3. Reversed Hazard Function

The mathematical formula for reversed hazard function is:

$$r(x) = \frac{f(x)}{F(x)}$$

So, we obtain the reversed hazard function of the BTM-I distribution as

$$r(x) = \frac{\sum_{k,l=0}^{\infty} w_{kl} \frac{p(k+l)}{\theta^{p(k+l)}} x^{p(k+l)-1}}{\frac{\left(\left(\frac{x}{\theta}\right)^p \left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right)^a}{a\,B(a,b)} . 2F1\left[a, 1-b, a+1; \left(\frac{x}{\theta}\right)^p \left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right]}$$

4. Odds Function

The mathematical formula for odds function is

$$O(x) = \frac{F(x)}{S(x)}$$

Again, using the binomial expansion of $\frac{\left(\left(\frac{x}{\theta}\right)^p \left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right)^{a+j}}{B(a,b)(a+j)}$,we have

$$f(x) = \sum_{k,l=0}^{\infty} w_{kl}\, g(x;\theta,p(k+l)), \quad 0 < x < \theta,$$

$$f(x) = \sum_{k,l=0}^{\infty} w_{kl} \frac{p(k+l)}{\theta^{p(k+l)}} x^{p(k+l)-1}, \quad 0 < x < \theta, \qquad (14)$$

The pdf of the i^{th} order statistic for the Beta transmuted Mukherjee-Islam distribution is given by

$$f_{i,n}(x) = \frac{1}{B(i,n-i+1)} f(x) \sum_{s=0}^{n-i} (-1)^s \binom{n-i}{s} [F(x)]^{i+s-1} = \frac{p}{B(i,n-i+1)} \left(\frac{1}{\theta}\right) \left(\sum_{k,l=0}^{\infty} w_{kl}\,(k+l) \left(\frac{x}{\theta}\right)^{p(k+l)-1} \right)$$

$$\sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \left[\sum_{k,l=0}^{\infty} w_{kl}\,(k+l) \left(\frac{x}{\theta}\right)^{p(k+l)-1} \right]^{i+s-1} .$$

Writing $u = \left(\frac{x}{\theta}\right)^p$, $f_{i,n}(x)$ can be expressed as

$$f_{i,n}(x) = \frac{p}{B(i,n-i+1)} \frac{1}{\theta} \left(\sum_{k,l=0}^{\infty} w_{kl}\,(k+l)\, u^{(k+l)-1} \right) *$$

$$\sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \left[\sum_{k,l=0}^{\infty} w_{kl}\,(k+l)\, u^{(k+l)-1} \right]^{i+s-1} , (17)$$

we note that in (17) we can write

$$\sum_{k,l=0}^{\infty} w_{kl}\; u^{(k+l)-1} = \sum_{m=0}^{\infty} w_m{}^*\, u^m .$$

And

$$\sum_{k,l=0}^{\infty} w_{kl}\,(k+l)\, u^{(k+l)-1} = \sum_{m=0}^{\infty} m\, w_m{}^*\, u^m .$$

$$M_x(t) = \frac{(k+l)pt^i\theta^i}{(i+(k+l)p)i!},$$

6. Entropy

An entropy gives a measure of variation of uncertainty for a random variable X with the pdf given by $f(x)$. The two most popular measures of entropy are Shannon entropy, (see Cordeiro and Nadarajah 2011) and Rényi entropy, (see Jones 2004). The Rényi entropy is defined by

$$I_R(v) = (1-v)^{-1} \log\left[\int_0^{\theta} f^v(x)dx\right].$$

Accordingly, Rényi Entropy can be expressed for Beta Transmuted Mukherjee-Islam distribution is defined by

$$M_x(t) = E(\mathbf{e^{tx}}) = \int_{-\infty}^{\infty} e^{t\,x} f(x)\; dx \;,$$

where $0 < x < \theta,\; p > 0,\; b > 0,\; a > 0,\; \theta > 0.$

So, we obtain the odds function of the BTM-I distribution as

$$O(x) = \frac{\frac{\left(\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right)^a}{a\,B(a,b)} .2F1\left[a,1-b,a+1;\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right.\right.\right.}{1-\frac{\left(\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right)^a}{a\,B(a,b)} .2F1\left[a,1-b,a+1;\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\right.\right.\right.} , \quad (2)$$

where $0 < x < \theta,\; p > 0,\; b > 0, a > 0,\; \theta > 0$.

Quantile's function is in wide spread use in statistics and often find representations in terms of lookup tables for key percentiles. The quantiles function of a distribution is the real solution of $F(x_q) = q$ for $0 \geq q \geq 1$.

The quantiles of the BTM-I distribution are obtained from cdf (9) as:

$$x_q = \theta\left[\left(1+\lambda + \sqrt{(1+\lambda)^2 - 4\lambda\left(I_u{}^{-1}[a,b]\right)}\right)/2\lambda\right]^{-\frac{1}{p}}, \quad (15)$$

where $I_u{}^{-1}(a,b)$ is the inverse of the incomplete Beta function and $u \sim U(0,1)$ with parameters a and b. The following expansion for the inverse of the Beta incomplete

and cdf is given by

$$F_{i,n}(x) = \sum_{k=l}^{n} \binom{n}{k} [F(x)]^k [1-F(x)]^{n-k},$$

$$= \int_0^{F(x)} \frac{1}{B(i,n-i+1)} [t]^{i-1}[1-t]^{n-i}.$$

Where $w_m{}^* = \sum_{k,l;k+l=m} w_{kl}$. Further, from Gradshteyn and Ryzhik (2000), for any positive integer r

$$\left(\sum_{k=0}^{\infty} a_k u^k\right)^r = \sum_{k=0}^{\infty} d_{r,k} u^k, \qquad (18)$$

where the coefficients $d_{r,k}$, for $k = 1,2,..,$ can be determined from the recurrence equation

$$d_{r,k} = (k_{a_0})^{-1} \sum_{m=1}^{k} [m(r+1)-k] a_m d_{r,k-m} \quad ,(19)$$

and $d_{r,0} = a_0{}^r$. Hence, $d_{r,k}$ comes directly from $d_{r,0}, \dots, d_{r,k-1}$ and, therefore, from $a_0, \dots, a_k$. Using (18) and (19) it follows that

$$f_{i,n}(x) = \frac{p}{B(i,n-i+1)} \frac{1}{\theta} * \left(\sum_{m=0}^{\infty} m\, w_m{}^* u^m\right)$$

$$* \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \left[\sum_{k=0}^{\infty} d_{i+s-1}, m^{u^m}\right]^{i+s-1}.$$

Where

$$d_{i+s-1,m} = (m\, w_0{}^*)^{-1}$$

$$* \sum_{q=1}^{k} \left([q(i+s)-m] w_m{}^* d_{i+s-1,m-q}\right),$$

$$d_{i+s-1,0} = (w_0{}^*)^{i+s-1}$$

$$= \left(\sum_{j=1}^{\infty} (-1)^j \binom{b-1}{j} \frac{\left(\left(\frac{x}{\theta}\right)^p \left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right)^{a+j}}{B(a,b)(a+j)}\right)^{i+s-1}$$

Combining terms, we obtain

$$f_{i,n}(x) = \frac{p}{B(i,n-i+1)} \frac{1}{\theta} *$$

$$\sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \sum_{m=l}^{\infty} \sum_{t=0}^{\infty} md_{i+s-1,t^{w^*},t^{w^*}m} u^{m+t},$$

3.2 Asymptotic Distribution of MLE

Since consistency is regarded as a weak positive property, and since in statistics one usually wants some idea of the distribution of the estimator, it is important to go beyond

$$I_R(v) = (1-v)^{-1}$$

$$* \log\left(\sum_{k,l=0}^{\infty} w_{kl}\right)^{v} \left(\frac{p^v \theta^{1+(-1+kp+lp)v} \left((k+l)\theta^{-((k+l)p)}\right)^{v}}{1+(-1+kp+lp)v}\right).$$

7. Order Statistics

Let $X_1, X_2, X_3, \dots, X_n$ be a simple random sample from the BTM-I distribution with cumulative distribution function (9) and probability density function (11). Let $X_1, X_2, X_3, \dots, X_n$ denote the order statistics from this sample. The pdf $f_{i,n}(x)$ of i^{th} order statistics $X_{(i)}$ is given by

$$f_{i,n}(x) = \frac{1}{B(i,n-i+1)} \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s}$$

$$* \sum_{m=l}^{\infty} \sum_{t=0}^{\infty} \frac{md_{i+s-1,t^{w^*},t^{w^*m}}}{m+t} \left(\frac{p(m+t)}{\theta} \left(\frac{x}{\theta}\right)^{p(m+t)-1}\right)$$

$$f_{i,n}(x) = \sum_{m=l}^{\infty} \sum_{t=0}^{\infty} c_i(m,t)\, g(x: \beta, (m+t)p),$$

where $g(x: \beta, (m+t)p)$ denotes the pdf of a Mukherjee-Islam distribution with parameter β and $(m+t)p$

$$c_i(m,t) = \frac{1}{B(i,n-i+1)} \left(\frac{mw_m{}^*}{(m+t)}\right) \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} d_{i+s-1,t}$$

3.Estimation and Inference on the Parameters of the Beta Transmuted Mukherjee-Islam distribution

3.1 Maximum Likelihood Estimation

The most general method of estimation is known as maximum likelihood (ML) estimators, which was initially formulated by Gauss. In the early 1920, Fisher firstly introduced ML as general method of estimation and later on developed by him in a

series of papers. He revealed the advantages of this method by showing that it yields sufficient estimators, which are asymptotically MVUES. Thus, the important feature of this method is that we look at the value of the random sample and then select our estimate of the

$$+(a-1)\sum_{i=1}^{n}\log\left(\frac{x}{\theta}\right)^{p} \qquad (21)$$

$$+(a-1)\sum_{i=1}^{n}\log\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^{p}\right)$$

$$+(b-1)\left(\sum_{i=1}^{n}\log\left[1-\left(\frac{x}{\theta}\right)^{p}\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^{p}\right)\right]\right)$$

We differentiate (21) with respect to $p, \theta, \lambda, a \text{ and } b$ respectively to obtain the elements of score vector $\frac{\partial(l(\varphi))}{\partial\varphi} = \left(\frac{\partial(l(\varphi))}{\partial p}, \frac{\partial(l(\varphi))}{\partial\theta}, \frac{\partial(l(\varphi))}{\partial\lambda}, \frac{\partial(l(\varphi))}{\partial a}, \frac{\partial(l(\varphi))}{\partial b}\right)$

$$x_q = \varphi\left[\left(1+\lambda + \sqrt{(1+\lambda)^2-4\lambda\left(I_u{}^{-1}[a,b]\right)}\right)/2\lambda\right]^{-\frac{1}{p}},$$

2. Set initial values for the parameters $(p_0, \theta_0\, \lambda_0, a_0, b_0)$.
3. Select a random sample from a BTM-I distribution $F(x; p, \theta, \lambda, a, b)$ of size n (say, 10, 30, 50, 75, 100 and 150).
4. Substitute the values obtained in Step (3) into the likelihood equations. Subsequently, solve these equations using the Newton-Raphson iterative method.
5. The solutions that we obtain in step (4) are the maximum likelihood estimators of $p, \theta, \lambda, a \text{ and } b$ say $(p_0, \theta_0\, \lambda_0, a_0, b_0)$ for the samples obtained in step (3).
6. Then compute the bias, mean squared errors and standard errors of the MLEs for the two thousand samples obtained in step (3). The standard errors (SE) are computed by inverting the observed information matrix. Bias and MSE are given by:

$$\boldsymbol{Bias(\hat{\varphi}) = (\hat{\varphi}_i - \varphi)},$$

$$MSE(\hat{\varphi}) = (\hat{\varphi}_i - \varphi)^2,$$

for $\varphi = (p, \theta, \lambda, a, b)$.

just consistency and look at results on limiting distributions. For results on limiting series of papers, he revealed the advantages of this method by showing that it yields succent estimators, which are asymptotically MVUES. For distributions we need more regularity conditions than what we need for consistency. Once you have a limiting distribution result, consistency will follow as a corollary to it. So, it makes sense that establishing limiting distribution results requires more regularity conditions than we need for consistency alone. The theorems are going to say that when all the regularity conditions hold, certain MLE-like estimates are going to be asymptotically normal. It is really important that you understand exactly what these theorems allow you to conclude. Let X_1, X_2 be iid observations from a density (pmf) $f(x|\varphi)$, where φ is a possibly vector valued parameter. Let $\varphi_0 \in \widehat{\varphi}$ denote the true value of the parameter. Under the Cram´ er-Rao conditions for asymptotic normality, there exists a sequence of roots $\hat{\varphi}_n$ of the unknown population parameter, the value of which the probability of getting the observed data is maximum. Let $x_1, x_2, x_3 \ldots x_n$ be a random sample from the Beta transmuted Mukherjee-Islam distribution with observed values $x_1, x_2, x_3 \ldots x_n$ and $\varphi = (p, \theta, \lambda, a, b)^T$ be parameter vector. The likelihood function for may be expressed as

$$L(\varphi) = \frac{\left(\frac{p}{\theta^p}\right)^n}{(B(a,b))^n}\prod_{i=1}^{n}x^{p-1}\left(1+\lambda-2\lambda\left(\frac{x}{\theta}\right)^p\right)$$

$$\left[\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right]^{a-1}\left[1-\left(\frac{x}{\theta}\right)^p\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^p\right)\right]^{b-1} \qquad (20)$$

The log-likelihood function is

$$L(\varphi) =$$

$$\log l(\varphi) = n\log p - np\log\theta - n\log B(a,b)$$

$$+(p-1)\sum_{i=1}^{n}\log x + \sum_{i=1}^{n}\log\left[1+\lambda-2\lambda\left(\frac{x}{\theta}\right)^p\right]$$

nominal confidence level (90%). A CP that is too low suggests the intervals are unreliable, while a CP that is too high indicates they are excessively wide and lack precision.
In contrast, Average Coverage Length (CL) measures the precision of the intervals. It represents the average width of the confidence intervals across all simulations. A smaller CL is desirable, as it signifies a more precise estimate.
Our analysis showed that as the sample size increased (specifically for sizes 100, 120, 150, 170, and 190) table (3.2), the Coverage Probability of the confidence intervals converged toward the nominal 95% level, indicating improved reliability. Simultaneously, the average coverage length decreased for these sizes, confirming that the intervals became more precise and efficient. This validates the fundamental relationship between larger sample sizes and computes the average estimates, along with biases, standard errors SE ($\widehat{\Theta}$) and MSE are given by:

$\boldsymbol{Bias(\hat{\varphi}) = \frac{\sum_{i=1}^{2000}(\hat{\varphi}_i - \varphi)}{2000}}$	(3.17)

$\boldsymbol{MSE(\hat{\varphi}) = \frac{\sum_{i=1}^{2000}(\hat{\varphi} - \varphi)^2}{2000}}$	(3.18)

for $\varphi = (p, \theta, \lambda, a, b)$.

7. Repeat steps (3), (4), (5) and (6), 2000 times (iteration) for n= 10, 30, 50, 75, 100 and 150, so computing $Bias(\hat{\varphi})$, $ASV(\hat{\varphi})$, $MSE(\hat{\varphi})$ and $SE(\hat{\varphi})$ for $\varphi = p, \theta, \lambda, a, b$ and n= 10, 30, 50, 80, 100 and 150.

where $\hat{\varphi} = (\hat{p}, \hat{\theta}, \hat{\lambda}, \hat{a}, \hat{b})$ is the maximum likelihood estimators of $\varphi = (p, \theta, \lambda, a, b)$.

To identify an effective estimator, examining both its bias and variance is crucial. While bias contributes less significantly to the mean squared error (MSE) compared to variance, both are considered. Our simulation runs were halted upon achieving consistent results across 2000 iterations.

likelihood equation which is consistent and which satisfies

$\sqrt{n}(\hat{\varphi}_n - \varphi_0) \rightrightarrows^L N(0, I^{-1}(\varphi_0))$.	(22)

where $\boldsymbol{I(\varphi)}$ is the Fisher information matrix

$$I = \begin{pmatrix} I_{pp} & I_{p\theta} & I_{p\lambda} & I_{pa} & I_{pb} \\ I_{\theta p} & I_{\theta\theta} & I_{\lambda\theta} & I_{\theta a} & I_{\theta b} \\ I_{\lambda p} & I_{\lambda\theta} & I_{\lambda\lambda} & I_{\lambda a} & I_{\lambda b} \\ I_{ap} & I_{a\theta} & I_{a3} & I_{aa} & I_{ab} \\ I_{bp} & I_{b\theta} & I_{\lambda b} & I_{ba} & I_{bb} \end{pmatrix}.$$

Where $I^{-1}(\varphi_0) = V$

Given that V is a function of certain parameters $(p, \theta, \lambda, a, b)$ (referenced as (22)), we estimate V, denoted as $\hat{V}$, by substituting these parameters with their corresponding Maximum Likelihood Estimates (MLEs).

The Simulation Study Steps

This section details a simulation study designed to assess how well maximum likelihood estimators (MLEs) perform with the Beta transmuted Mukherjee-Islam (BTM-I) distribution. Our evaluation of the MLEs behavior in finite samples relies on examining specific characteristics:

1. To initiate the simulation, we'll generate two thousand samples, each of size *n*, from the BTM-I distribution. This will be done using the inversion method to produce the necessary values:

3.3 Results of the Simulation of the Parameters BTM-I Distribution

In this study, we evaluated the performance of confidence intervals for parameters using two key metrics: Coverage Probability (CP) and Average Coverage Length (CL).
Coverage Probability (CP) serves as a measure of the reliability of confidence intervals. It represents the percentage of times a calculated interval contains the true parameter value. The optimal goal is for the empirical CP to be very close to the

follow-up at Sana'a Prosthetics and Physical Therapy Center.

3.5, 5.5, 4 , 5 , 2.5, 7 , 5.5, 3.5, 6.5, 4.5, 1.5, 5.5, 3.5, 6 , 4.5, 7 , 5 , 3, 6.5, 4.1,2 , 5 , 3.5, 5 , 4.5, 3 , 5 , 3.5 , 6 , 4 , 3 , 6 , 4.5, 4, 5.5, 2 , 6.5, 3 , 6, 4.5, 2.5,5.5, 4 , 5 , 3 , 1.5, 6 , 4.5, 2.5, 3.5, 3.5, 6 , 4 , 7 , 4 , 2.5, 6 , 3.5, 5.5, 4.5, 4.5, 7 ,6 , 3 , 5.5, 2.5, 6.5, 3.5, 5.5, 4.5, 2.5, 5.5, 3.5, 6.5, 4.5, 7 , 5.5, 2.5, 6.5, 4.5, 1.5, 6.5, 4.5,7 , 5.5, 2.5, 6.5, 3.5, 5.5, 4.5, 3.5, 5 , 4 , 6 , 5 , 1.5, 6 , 3.5, 5 , 4 , 2.5, 5 , 4 , 7 ,4.5, 1.5, 6.5, 3 , 5 , 4 , 3 , 5 , 4.5, 6.5, 3.4, 2 , 6.5, 3.5, 5 , 4 , 4 , 6.5, 5 , 7 , 4.8,1.4, 6.2, 3.7, 5.7, 4.6, 2.5, 5.4, 3.9, 6.7, 4.3, 7.0, 5.0, 2.9, 6.0, 4.1, 3.8, 6.8, 5.6, 7.0, 5.5, 2.4,6.9, 3 , 5.5, 4.5, 2 , 5.1, 4 , 6.5, 4 , 5, 4.2, 3.5, 6 , 4.5, 1.5, 6.5, 5.3, 7 , 4.2, 1.7, 6.1,3.6, 5.8, 4.4, 3.2, 5.0, 4.5, 6.8, 5.0, 2.6, 6.6, 3.9, 5.4, 4.1, 2.8, 5.7, 4.3, 6.5, 4.7, 7.0, 5.5, 3.4,6.2, 4.9, 3.6, 6 , 5.1, 7 , 5.9, 2.1, 6.8, 3 , 5.3, 4.6 .

to illustrate the importance and potentiality of the BTM-I distribution and some of the models generated from Mukherjee-Islam distributions. The goodness-of-fit statistic for this distribution and other competitive distributions are compared and the MLEs of themselves. We delved deep into the statistical properties of the BTM-I distribution, meticulously deriving key characteristics such as its quantile function, various moments, the moment generating function, measures of mean and median deviations, and the Rényi entropy. We also precisely formulated its probability density function (pdf), cumulative distribution function (cdf), and hazard function, visually illustrating their behaviors in Figures 2.1 and 2.2.

To estimate the BTM-I parameters, we used the widely recognized Maximum Likelihood Estimation (MLE) technique. Our extensive simulation studies showed highly satisfactory results: the MLEs for the BTM-I parameters demonstrated minimal bias, appearing nearly unbiased across all scenarios. As expected from the asymptotic properties of MLE, the mean squared error (MSE) consistently decreased with increasing sample sizes, confirming the estimators' desirable behavior.
Furthermore, our study showed that as the sample size increased, the Coverage Probability (CP) of the confidence intervals converged towards the nominal 0.90, 0.95 and 0.99 levels, indicating

their parameters are provided. Models' comparisons entailed the consideration of various criteria such as the maximized log-likelihood (-2ℓ), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) for the data set. Minimum values rule of AIC, BIC, CAIC and HQIC is taken into consideration for selecting the best model to fit. These statistics are given by

$$AIC = 2K - 2l(\hat{\varphi})$$

$$CAIC = AIC + \frac{2k(k+1)}{n-k-1}$$

$$BIC = klog(n) - 2l(\hat{\varphi})$$

$$HQIC = 2klog\big(log(n)\big) - 2l(\hat{\varphi}),$$

where K is the number of parameters in the statistical model, n is the sample size and $l(\hat{\varphi})$ is the log-likelihood function evaluated at (21), the maximum likelihood estimates, φ is the parameters. These models include:

- Mukherjee-Islam (M-I)

$$f(x) = \frac{p}{\theta^p} x^{p-1}.$$

- Transmuted Mukherjee-Islam (TM-I)

$$f(x) = \frac{p}{\theta^p} x^{p-1} \left(1 + \lambda - 2\lambda \left(\frac{x}{\theta}\right)^p\right).$$

- Beta Mukherjee-Islam (BM-I)

$$f(x) = \frac{\frac{p}{\theta^p} x^{p-1}}{B(a,b)} \left[\left(\frac{x}{\theta}\right)^p\right]^{a-1} \left[1 - \left(\frac{x}{\theta}\right)^p\right]^{b-1}.$$

reliability. Concurrently, the Average Coverage Length (CL) decreased,

and enhanced reliability and precision in interval estimation.

4. Application

In this section, we provide application to real data set on the duration of lower limb prosthesis lifespan (in years) for a sample of 200 individuals receiving treatment and efficient software libraries in languages like Python or R to streamline data modeling and statistical inference. Furthermore, exploring the use of parallel computing or machine learning-based approaches could lead to faster parameter estimation, especially when handling very large datasets or conducting extensive simulations. Finally, research could extend to studying multivariate extensions based on the Beta Transmuted Mukerjee-Islam distribution, which would enable modeling complex relationships between several variables simultaneously and enhance the distribution's ability to improve address more intricate statistical phenomena, demonstrating enhanced precision and efficiency of the estimations.

The empirical efficacy and clear superiority of the BTM-I distribution were rigorously established through its application to three distinct real-world datasets. A comprehensive comparative analysis, using goodness-of-fit statistics and information criteria (AIC, BIC, HQIC, and CAIC), consistently and unequivocally revealed the BTM-I model's significant advantage in data representation. As detailed in Table 4.3, these findings, corroborated by higher log-likelihood values, overwhelmingly substantiate that the BTM-I distribution provides a statistically superior fit to empirical data compared to its sub-distributions and other competing models. This robust evidence firmly positions the BTM-I distribution as a highly promising and more appropriate statistical tool for accurately modeling diverse real-world phenomena.

5.1 Recommendations and further Research

Building on the proposed distribution, there are many fruitful avenues for future research. Researchers can explore methods for parameter estimation based on various types of incomplete data, such

address more intricate statistical phenomena

4.1 Results and Conclusion

The results unequivocally demonstrate that the BTM-I distribution significantly outperforms its sub-models in fitting both datasets. The BTMI model obtained the lowest values for all goodness-of-fit criteria ($\hat{l}$, AIC, BIC, CAIC, and HQIC) when compared to the TM-I and M models. This finding proves that the additional parameters in the BTM-I model enhance its flexibility and ability to capture complex data characteristics more effectively. Therefore, based on this evidence, the BTM-I distribution is selected as the best model for analyzing both the prosthesis lifespan and loan grace period data.

5. Conclusions and Recommendations

We have introduced and thoroughly investigated a novel statistical distribution generator: the Beta Transmuted Mukherjee-Islam (BTM-I) distribution. This new generator is highly versatile, giving rise to several important sub-models, including the Transmuted Mukherjee-Islam, Beta Mukherjee-Islam, and Mukherjee-Islam distributions as truncated data or grouped data, which opens new horizons for broader practical applications. Additionally, it's crucial to study the robustness properties of the proposed distribution and its estimators when dealing with outliers or deviations from standard assumptions. This will ensure the reliability of results in real-world scenarios. Developing and applying goodness-of-fit tests specifically tailored for this complex distribution would also allow for more accurate assessment of its suitability for empirical data.

From a computational perspective, the complexity of the proposed distribution offers an excellent opportunity to develop optimized computational algorithms. This includes designing and implementing

Table (3.1): Estimated parameters, Bias, SE, and MSE of the BTM-I distribution

		10	20	30	50	75	100	150
Parameters Estimation	$\hat{p}$	1.676303	1.671868	1.670474	1.668979	1.669154	1.669022	1.669165
	$\hat{\theta}$	1.908484	1.915158	1.915691	1.9155828	1.913456	1.912676	1.910694
	$\hat{\lambda}$	0.688209	0.681034	0.679513	0.6787709	0.679441	0.679657	0.680622
	$\hat{a}$	0.524008	0.536262	0.538139	0.540078	0.537622	0.536719	0.534964
	$\hat{b}$	3.210907	3.217192	3.218944	3.2203421	3.220591	3.220916	3.221109
Bias	p	-0.03137	-0.015907	-0.010651	-0.00642	-0.004278	-0.00321	-0.00214
	θ	0.130848	0.065758	0.043857	0.0263117	0.017513	0.013127	0.008738
	λ	0.018821	0.009052	0.005984	0.0035754	0.002393	0.001797	0.001204
	a	-0.077599	-0.038187	-0.025395	-0.015198	-0.010165	-0.00763	-0.0051
	b	0.211091	0.105859	0.070632	0.0424068	0.028275	0.021209	0.014141
SE	p	0.135318	0.094807	0.078165	0.061068	0.050742	0.044187	0.036436
	θ	0.066067	0.048268	0.040034	0.031301	0.025843	0.022459	0.018432
	λ	0.408490	0.289577	0.236973	0.1838197	0.150414	0.130349	0.018432
	a	0.162995	0.117931	0.096624	0.0751073	0.061035	0.052766	0.106511
	b	0.523293	0.372414	0.304446	0.2359488	0.192397	0.166564	0.135846
MSE	p	0.000994	0.000254	0.000114	0.0000413	0.000018	0.000010	4.58E-06
	θ	0.017139	0.004327	0.001924	0.0006925	0.000307	0.000172	0.000076
	λ	0.000366	0.000084	0.000036	0.0000129	5.77E-06	3.25E-06	1.46E-06
	a	0.006069	0.001461	0.000646	0.0002312	0.000103	0.000058	0.000026
	b	0.044578	0.011207	0.004989	0.0017984	0.000799	0.000449	0.0002

Table 3.2: 90% Confidence Interval Performance: Coverage Probabilities and Average Lengths by Sample Size

parameter		90	100	120	150	170	190
p	CP	0.814	0.973	0.946	0.961	0.98	0.9831
	CL	0.2199576	0.1985256	0.1944961	0.1760923	0.1647812	0.1545779
θ	CP	0.7455	0.9015	0.9255	0.9434	0.952	0.963
	CL	0.0938394	0.0860413	0.0824891	0.0772806	0.0727578	0.0690554

λ	CP	0	0.354	0.725	0.891	0.89	0.91
	CL	0.4951102	0.4527949	0.4356496	0.4055485	0.3811945	0.361018
a	CP	0.741	0.8994	0.90	0.9254	0.9305	0.9358
	CL	0.4816498	0.4424668	0.4231558	0.3974993	0.3740586	0.3547786
b	CP	0.735	0.898	0.924	0.9425	0.938	0.8995
	CL	0.4764171	0.4373207	0.4231558	0.3926253	0.3693473	0.3500467

Table (4.1) displays the estimated parameters and their standard errors for the first real data set

	$\hat{p}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
BTM-I	1.6101767	4.4556021	0.677199	0.7584412	1.0667177
SE	9.3142E-7	0.0003716	1.3021E-6	0.0000271	0.0000101
BM-I	1.61203	5.81034	-	0.80101	1.98544
SE	2.0732E-6	0.0005063	-	0.0000122	0.0000117
TM-I	2.670101	9.901	0.940301	-	-
SE	0.0000148	0.002021	7.2981E-6	-	-
M-I	3.10152	7.19201	-	-	-
SE	0.0000732	0.0007818	-	-	-

Table (4.2): Descriptive statistics for the first real data set

N	Mean	Median	SD	Variance	Skewness	Kurtosis	Minimum	Maximum
200	4.6235	4.5	0.10562	2.231	0.172	0.342	1.4	7

Table (4.3): Log-likelihood and information criteria for the first real data set

	$2\hat{l}$	AIC	BIC	CIAC	HQIC	Ranks
BTM-I	106.35219	116.35219	132.84378	116.66147	123.02608	**1**
BM-I	195.81061	203.81061	217.00388	204.01574	209.14972	**2**
TM-I	431.1521	437.1521	447.04705	437.27455	441.15644	**3**
M-I	765.1831	769.1831	775.7797	769.24398	771.85262	**4**

References

أولاً: المراجع باللغة العربية:

[1] هرمز، أمير حنا. (1990). الإحصاء الرياضي، جامعة الموصل، العراق: مديرية دار الكتب للطباعة والنشر.

[2] الصياد، جلال مصطفي. (1993). الاستدلال الإحصائي، الرياض، المملكة السعودية العربية: دار المريخ.

Secondly: References in English

[1] Afify, A., Yousof, H., and Nadarajah, S. (2017). The beta transmuted-H family for lifetime data. Statistics and its Interface, 10(3), 505–520.

[2] Agresti, A. (2015). Foundations of Linear and Generalized Linear Models. Wiley.

[3] Akaike, H. (1974). A new look at the statistical model identification. IEEE Transactions on Automatic Control, 19(6), 716–723.

[4] Al-Aziz, S. H., & Al-Marzooq, A. (2022). The Beta Transmuted Gumbel Distribution: Theory and Applications. Journal of Statistics Applications and Probability, 11(2), 345–359.

[5] Al-Hamadi, A. M. (2022). A new generalized transmuted Weibull distribution. Journal of Mathematical Sciences and Applications, 12(1), 1–12.

[6] Al-Maqtari, S. S., & Al-Ameri, H. M. (2023). A new generalized transmuted distribution with application to engineering data. Journal of Modern Applied Statistical Methods, 22(1), 1–18.

[7] Al-Mashreqi, A. A., & Al-Dhaheri, F. K. (2023). A new transmuted Pareto distribution for modeling financial data. Journal of Risk and Financial Management, 16(5), 1–15.

[8] Al-Mutairi, A. M., & Al-Shehri, A. A. (2023). A new transmuted log-logistic distribution with application to survival data. Journal of the Saudi Society for Statistical Sciences, 10(2), 145–160.

[9] Al-Qadhi, F. M., & Al-Jabri, H. A. (2023). The Beta Transmuted Exponential Distribution: Theory and Applications. Journal of Statistical Research, 57(1), 5–20.

[10] Al-Zoubi, A. M., & Al-Amri, M. N. (2024). The Beta Transmuted Kumaraswamy Distribution: Theory and Applications. Journal of Mathematical and Statistical Sciences, 12(2), 1–18.

[11] Anderson, T. W. (2003). An Introduction to Multivariate Statistical Analysis (3rd ed.). Wiley.

[12] Arif, M., & Siddiqui, S. A. (2024). A new generalized Mukherjee-Islam distribution. Journal of Statistics Applications and Probability, 13(2), 1–15.

[13] Aryal, G. R., and Tsokos, C. P. (2009). On the transmuted extreme value distribution with application. Nonlinear Analysis: Theory, Methods and Application, 71(12), 1401–1407.

[14] Ashour, S. K., and Eltehiwy, M. A. (2013). Transmuted Lomax Distribution. American Journal of Applied Mathematics and Statistics, 1(6), 121–127.

[15] Aslam, M., & Ashraf, S. (2024). The Beta Transmuted Dagum Distribution: A new generalization with applications. Journal of Statistical Sciences, 19(1), 1–12.

[16] Buckley, C., & Shaw, A. (2009). The business of climate change: A perspective from the private sector. **Journal of Business Ethics**, 88(4), 743-759.

[17] Bae, S., & Kim, J. (2024). A new generalized transmuted distribution for modeling survival data. Journal of Statistical Computation and Simulation, 94(5), 899–915.

[18] Berger, J. O. (1985). Statistical Decision Theory and Bayesian Analysis (2nd ed.). Springer.

[19] Billingsley, P. (1995). Probability and Measure (3rd ed.). Wiley. DiCiccio, T. J., and Efron, B. (1996). Bootstrap confidence intervals. Statistical Science, 11(3), 189–212.

[20] Bourguignon, M., Ghosh, I., and Cordeiro, G. M. (2015). General results for the transmuted family of distributions and new models. Journal of Probability and Statistics, 2016, 1–12.

[21] Box, G. E. P., Jenkins, G. M., Reinsel, G. C., & Ljung, G. M. (2015). Time Series Analysis: Forecasting and Control (5th ed.). Wiley.

[22] Bozdogan, H. (1987). Model selection and Akaike's information criterion (AIC): The general theory and its analytical extensions. Psychometrika, 52(3), 345-370.

[23] Casella, G., and Berger, R. L. (2002). Statistical Inference (2nd ed.). Duxbury Press.

[24] Cheng, R. C. H., and Iles, T. C. (1983). Confidence bands for cumulative distribution functions of continuous random variables. Technometrics, 25(1), 77–85.

[25] Cheng, R. C. H., and Iles, T. C. (1988). One-sided confidence intervals for the parameters of a normal and extreme value distribution. Communications in Statistics - Simulation and Computation, 17(2), 643–651.

[26] Chhetri, S., Long, H., and Aryal, G. (2017). The Beta Transmuted Pareto Distribution: Theory and Applications. Journal of Statistics Applications and Probability, 6(2), 243–258.

[27] Cordeiro, G. M., & Nadarajah, S. (2012). Beta generalized distributions. International Journal of Statistics and Probability, 1(1), 1–12.

[28] Cordeiro, G. M., & Nadarajah, S. (2013). The beta transmuted family of distributions. Journal of Statistical Computation and Simulation, 83(2), 241–264.

[29] Cordeiro, G. M., and Nadarajah, S. (2011). Closed form expressions of moments of a class of a beta generalized distribution. Brazilian Journal of Probability and Statistics, 25, 14–33.

Bayesian Data Analysis (3rd ed.). Chapman and Hall/CRC.

[47] Glynn, P. W., & Whitt, W. (1992). Limiting distributions of high-level excursions of a random walk. Advances in Applied Probability, 24(1), 74–101.

[48] Gomes, L., & Vasconcelos, A. (2023). A new generalized transmuted distribution for modeling financial data. Journal of Risk and Financial Management, 16(6), 1–15.

[49] Greene, W. H. (2018). Econometric Analysis. Pearson.

[50] Gupta, R. D., & Kundu, D. (1999). A new generalized exponential distribution. Computational Statistics & Data Analysis, 34(3), 323–332.

[51] Gupta, R. D., & Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distributions. Biometrical Journal, 43(1), 117–130.

[52] Hamilton, J. D. (1994). Time Series Analysis. Princeton University Press. Azzalini, A. (1996). Statistical Inference Based on the Likelihood. Chapman & Hall.

[53] Hannan, E. J., & Quinn, B. G. (1979). The determination of the order of an autoregression. Journal of the Royal Statistical Society: Series B (Methodological), 41(2), 190-195.

[54] Hansen, B. E. (2022). Econometrics. Princeton University Press.

[55] Hastie, T., Tibshirani, R., & Friedman, J. (2009). The Elements of Statistical Learning: Data Mining, Inference, and Prediction (2nd ed.). Springer.

[56] Hinkley, D. V. (1978). Likelihood inference about location and scale parameters. Biometrika, 65(3), 467–474.

[57] Hurairah, A. (2019). The Beta Transmuted Gumbel Distribution: Theory and Application. International Journal of Scientific Research in Mathematical and Statistical Sciences, 6(6), 62–73.

[58] Hurairah, A. (2020). The Beta Transmuted Dagum Distribution: Theory and Applications. Research in Mathematical and Statistical Sciences, 8(2), 5–11.

[59] Hurairah, A., and Alabid, A. (2020). The Beta transmuted Lomax distribution with applications. Statistics in Transition New Series, 21(2), 13–34.

[60] Hurvich, C. M., & Tsai, C. L. (1989). Regression and time series model selection in small samples. Biometrika, 76(2), 297–307.

[61] Ibragimov, I. A., & Has'minskii, R. Z. (1981). Statistical Estimation: Asymptotic Theory. Springer-Verlag.

[62] Ibragimov, I. A., & Has'minskii, R. Z. (1981). Statistical Estimation: Asymptotic Theory. Springer-Verlag.

[63] John Wiley & Sons. Pal, M., and

[30] Cox, D. R., & Oakes, D. (1984). Analysis of Survival Data. Chapman and Hall/CRC.

[31] DasGupta, A. (2008). Asymptotic Theory of Statistics and Probability. Springer.

[32] Davison, A. C. (2003). Statistical Models. Cambridge University Press.

[33] Deka, D., Das, B., and Baruah, B. K. (2017). Transmuted Exponentiated Gumbel Distribution (TEGD) and its Application to Water Quality Data. Pakistan Journal of Statistics and Operation Research, 13(1), 115–126.

[34] Dobson, A. J., & Barnett, A. G. (2018). An Introduction to Generalized Linear Models (4th ed.). Chapman and Hall/CRC.

[35] Doganaksoy, N., and Schmee, J. (1991). The accuracy of asymptotic confidence intervals for the parameters of the smallest extreme value and normal distributions. IEEE Transactions on Reliability, 40(3), 329–333.

[36] Doganaksoy, N., and Schmee, J. (1993). Evaluation of asymptotic normal intervals and likelihood ratio-based intervals for the smallest extreme value and normal distributions under various degrees of censoring. IEEE Transactions on Reliability, 42(3), 442–446.

[37] Efron, B. (1987). Better bootstrap confidence intervals. Journal of the American Statistical Association, 82(397), 171–185.

[38] Elbatal, I., & Aryal, G. (2016). The transmuted Mukherjee-Islam distribution. Journal of Applied Statistics, 43(11), 1957–1972.

[39] Elbatal, I., & Aryal, G. (2017). The transmuted Gumbel distribution with applications. Journal of Statistics Applications and Probability, 6(1), 21–36.

[40] Elbatal, I., & Aryal, G. (2018). The transmuted Lindley distribution with applications. Journal of Applied Statistics, 45(1), 1–16.

[41] Elbatal, I., & Aryal, G. (2019). The Beta Transmuted Lomax Distribution. Journal of Statistical Research, 53(1), 21–36.

[42] Elbatal, I., and Aryal, G. (2015). Transmuted Dagum distribution with applications. Chilean Journal of Statistics, 6(2), 31–45.

[43] Ethier, S. N., & Kurtz, T. G. (1986). Markov Processes: Characterization and Convergence. Wiley.

[44] Eugene, N., Lee, C., and Famoye, F. (2002). Beta-normal Distribution and its Applications. Communications in Statistics – Theory and Methods, 31(4), 497–512.

[45] Ferguson, T. S. (1996). A Course in Large Sample Theory. Chapman & Hall/CRC. Fu, M. C., & Hu, J. Q. (1997). Conditional Monte Carlo: Applications to Stochastic Process Simulation. Kluwer Academic Publishers.

[46] Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A., & Rubin, D. B. (2013).

[68] Lawless, J. F. (1975). Construction of confidence intervals for parameters of the Weibull and extreme value distributions. Technometrics, 17(1), 113–118.
[69] Lawless, J. F. (1978). Confidence interval estimation for the parameters of the Weibull and extreme value distributions. Technometrics, 20(3), 355–360.
[64] Transmuted Weibull Distribution. Austrian Journal of Statistics, 43(2), 133–149.
[65] Jones, M. C. (2004). Family of Distributions Arising from Distribution of Order Statistics, 13, 1–43.
[66] Kundu, D., & Raqab, M. Z. (2005). The transmuted Weibull distribution. Communications in Statistics - Theory and Methods, 34(3), 737–753.
[67] Law, A. M. (2015). Simulation Modeling and Analysis. McGraw-Hill.

Figure 1: Probability Density Functions (PDFs) of the BTM-I distribution for different parameter values.

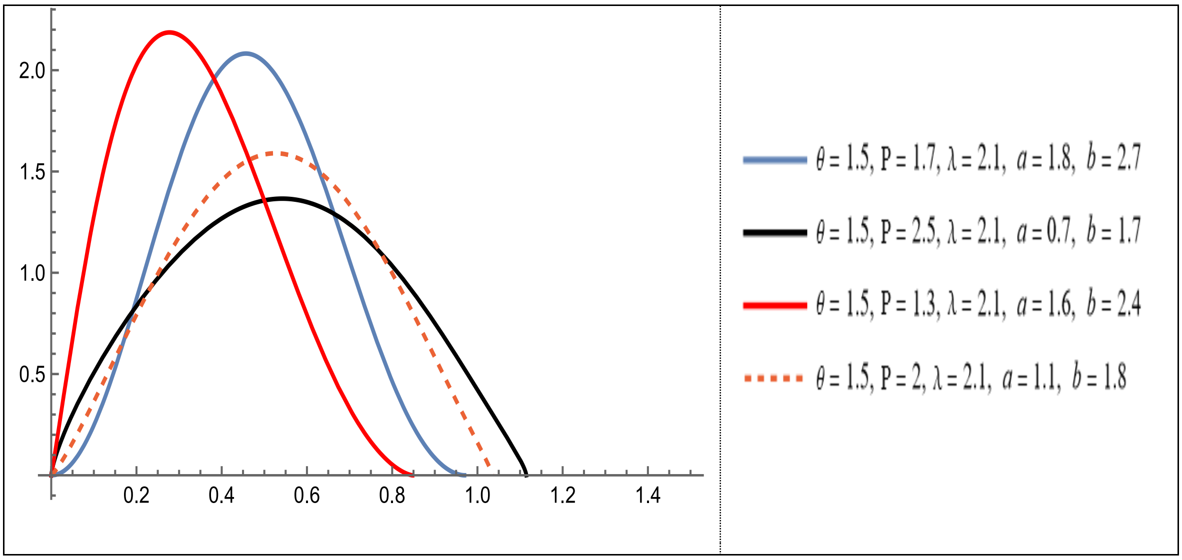

Figure 2: The hazard rate function of the BTM-I distribution for different parameter values.