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The Transmuted Kumaraswamy Burr III Distribution: Estimation and Applications

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Keywords

1. Burr III distribution
2. Rank Transmutation Map
3. Asymptotic Theory
4. Transmuted Kumaraswamy Burr III distribution
5. Maximum likelihood estimation
6. Information Criterion

Abstract:

This study introduces a new probability distribution referred to as the Transmuted Kumaraswamy Burr III distribution (KwB III) using the Quadratic Rank Transmuted Map (QRTM) proposed by Shaw et al. (2007). The statistical properties of the distribution were thoroughly invested, including the moments, moment generating function, mean, quantiles and random number generator, Rényi entropy, order statistics and the reliability analysis. A simulation study was carried out to evaluate the performance of the MLEs, and showed that the estimators were nearly unbiased and that the mean squared error (MSE) decreased with increasing sample size, confirming efficiency and consistency.

In addition, the study applied several information criteria, such as -2 log-likelihood ($-2\hat{l}$), the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), corrected Akaike Information Criterion (AICc), and Hannan-Quinn information criterion (HQIC) to evaluate model fit and compare the proposed distribution with existing alternatives. Overall, findings demonstrate that the proposed distribution is flexible, statistically robust, and suitable for modelling various types of data, especially when classical models are inadequate.

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توزيع كوماراسوامي بيور النوع الثالث المحول: التقدير والتطبيقات

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الكلمات المفتاحية

1. توزيع بيور النوع الثالث (B III)
2. خريطة تحويل الرتبة (RTM)
3. نظرية التقارب
4. توزيع كوماراسوامي(*TKwB III*)
5. طريقة الإمكان الأعظم (MLE)
6. معايير جودة المطابقة (IC)

الملخص:

في هذه الدراسة، تم تقديم توزيع احتمالي جديد يعرف بـ توزيع كوماراسوامي بيور النوع الثالث المحول The Transmuted Kumaraswamy Burr III distribution (KwB III)، والذي تم إنشاؤه من خلال تعميم توزيع كوماراسوامي بيور النوع الثالث باستخدام خارطة تحويل الرتب التربيعية. وقد تمت دراسة خصائصه الإحصائية بشكل شامل، بما في ذلك العزوم، الدالة المولدة للعزوم، المتوسط، الكميات ومولد الأرقام العشوائية معكوس الدالة التراكمية، وانتروبيا ريني، الإحصاءات المرتبة وتحليل الموثوقية ودالة البقاء. تم تقدير معلمات هذا التوزيع باستخدام طريقة الإمكان الأعظم. أجريت دراسة محاكاة لتقييم أداء المقدرات باستخدام برنامج (SAS 9.4) باستخدام احجام مختلفة للعينات لتقييم أداء المقدرات. وأظهرت النتائج أن مقدرات الإمكان الأعظم كانت تقريباً غير متحيزة، وأن متوسط مربعات الخطاء (MES) تناقص مع زيادة حجم العين، مما يدل على كفاءتها واتساقها. تم تطبيق مجموعة من البيانات الحقيقية على توزيع كوماراسوامي بيور النوع الثالث المحول (TKwB III) بالإضافة الي التوزيعات المشابهة، بهدف تقييم جودة ومرونة هذا النموذج في تحليل البيانات. وقد تم الاعتماد على عدد من المقاييس الإحصائية، منها: -2ℓ ، AICc, AIC، BIC وHQIC. للمقارنة بين أداء النماذج المختلفة. وقد أظهرت نتائج التحليل والمحاكاة أن نموذج كوماراسوامي بيور النوع الثالث المحول (TKwB III) يتمتع بمرونة وكفاءة جيدة في التقدير، حيث سجل أقل القيم للمعايير الإحصائية مقارنة بالتوزيعات الأخرى المستخدمة في الدراسة. وتشير هذه النتائج الي أن نموذج كوماراسوامي بيور النوع الثالث المحول (TKwB III) قادر علي تمثيل البيانات.

Introduction:

The probability distributions are used in many areas of life such as engineering, medicine, economics, and others. The application of statistical tools depends on the underlying probability model of the data. Generalized probability distributions are very common practice in the theory of statistics. In order to generalize probability distributions various distributions are proposed in literature which add extra parameters to existing baseline probability distributions so that the new distribution can increase the flexibility of the distributions to capture the complexity of the data. Several generalized (or G) classes are available in literature, Exponentiated Family, Marshall-Olkin Family, Beta-G Family, Kumaraswamy-G Family, etc. Many generalized distributions have been developed over the past few decades for modeling data in several areas. Shaw and Buckley (2007), for example, proposed a new approach of constructing and generalizing statistical distributions. Their concept is defined by transmuting maps that are functional composition of the cumulative distribution function of one distribution with the quantile function of another distribution. One of the maps is referred to as the Quadratic Rank Transmutation Map (QRTM). Based on Aryal and Tsokos's (2009) work on the transmuted extreme value distribution, a number of transmuted families of distributions have been proposed and discussed in the existing literature. For example, Transmuted-G Family of Distributions and Properties and Applications were proposed by Afify (2016); Transmuted Kumaraswamy distribution by Khan and King (2016); Transmuted Exponentiated Kumaraswamy Distribution by Joseph and Ravindran, (2023); The Kumaraswamy Transmuted Pareto by Chhertri et al. (2017).
In this study, a generalization of the Kumaraswamy distribution by using the (QRTM) referred to as the Transmuted Kumaraswamy Burr III distribution (TKwBIII) is proposed. Explicit expressions are derived for the moments, moment generating function, quantile function, Rényi entropy, and order statistics. Survival analysis is also performed. We discuss estimation of the distribution parameters by the method of maximum likelihood, and the Fisher information matrix is derived. Simulation of random variables is performed in order to investigate the performance of the estimates. A real data set is used to compare the new distribution with its sup-model distributions.

2. Statement of the Problem:

Throughout relevant literature, there exist many lifetime distributions that have no enough flexibility; therefore, researchers often resort to new generalizations for classical distributions through quadratic rank transmutation map. In this study, a generalization of the Burr type III distribution has been proposed using the Kumaraswamy distribution called the Kumaraswamy-Burr III distribution, and then we introduced a new generalization to the Kumaraswamy distribution using the transmutation map approach proposed by Shaw and Buckley (2007). The new distribution which generalizes the Kumaraswamy-Burr III distribution is called the Transmuted Kumaraswamy-Burr III distribution.

3. Objectives of the Study:

This study is expected to achieve the following objectives:

1. To propose a new generalization by using the Kumaraswamy distribution called the Transmuted Kumaraswamy-Burr III distribution.
2. To discuss and study some mathematical and statistical properties of the new distribution.
3. To investigate the Maximum likelihood estimators for the parameters of the TKwBIII distribution, and observed information matrix is derived.
4. To investigate the properties of Maximum likelihood estimators such as Bias, mean square error (MSE), stander error (SE), based on simulation using (SAS) software.
5. To investigate the confidence intervals for the parameters of the new distribution.
6. To illustrate the application of the new distribution using comparing a real data set with the sub-models to provide the flexibility and efficiency of the new (TKwB III) distribution.

4. Significance of the Study:

Statistical probability distributions are the foundation of statistical methodology in both theory and practice. They are also the center for all the statistical methods including inference, modeling, survival analysis, reliability analysis, etc. Statistical distributions have been used in modeling extreme events such as waiting time, floods, and so forth. The knowledge of the appropriate distribution of real data sets greatly improves the sensitivity, power and efficiency of the statistical test associated with the data sets, and hence, the development of the new generators for modifying existing distributions to improve their goodness of-fit is obligatory.

The significance of this study stems from the fact that a new generalization is proposed using the transmuting map called Transmuted Kumaraswamy-Burr III Distribution, a distribution that is expected to have a great flexibility and efficiency through introducing additional parameters by using transmuted Kumaraswamy and transmuted map.

5.**Basic Concepts:**

This section presents some basic concepts of the new distribution that can help achieve the objectives of the new distribution. These concepts include transmutation maps, the Quadratic Rank Transmutation Map, asymptotic theory, and simulation in statistics.

5.1Transmutation Maps:

Classical probability function often falls short of capturing complex data patterns, and therefore there is a real need for developing new flexible probability distributions that can improve the quality of statistical analysis. Toward this end, considerable attention has recently been given to extension and generalization.

5.2 Quadratic Rank Transmutation Map:

various authors have developed several probability distributions using the QRTM for several choices of baseline cdf $(G(x))$. The quadratic rank transmutation map is part of the rank transmutation maps. Aryal and Tsokos (2009) presented transmuted generalized extreme value distribution and obtained its various mathematical properties. Khan and King proposed Transmuted Modified Weibull and studied some of its statistical properties.

The cumulative distribution function (cdf) of

$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, \lvert\lambda\rvert \leq 1,$	(1)

the QRTM has the following formula:
And the pdf is given as:

$f(x) = g(x)[(1+\lambda) - 2\lambda G(x)].$	(2)

where $G(x)$ is the (cdf) of baseline distribution.

5.3 Asymptotic Theory:

Apart from a few classical results in mathematical statistics, determining the precise distributional properties of a statistic or other random variables can often be quite challenging. Having a reliable approximation of the exact distribution is extremely beneficial for examining the characteristics of various statistical methods.

5.4 Simulation in Statistics:

Simulation is a tool to evaluate the performance of a system that is already existing or is proposed under different arrangement of advantage and over long periods of real time. Simulation involves replicating the functioning of a real-world process or system over a period of time. It entails creating a synthetic history of a system and analyzing that history to make inferences about the operational characteristics of the actual system it represents.

6. The structure of the new distribution The Transmuted Kumaraswamy Burr III distribution:

The Transmuted Kumaraswamy Burr III distribution's sub-models play a crucial role in reliability and survival analysis. These include:

6.1 The Kumaraswamy distribution:

The Kumaraswamy probability distribution was proposed by Poondi Kumaraswamy (1980) for double bounded random processes for hydrological applications.

The Kumaraswamy (1980) distribution is similar to Beta distribution, but it is easy to use, especially in simulation studies. The Kumaraswamy double bounded distribution denoted by Kw (a, b) distribution is a family of continuous probability distributions defined on the interval [0,1], with cumulative distribution function (cdf) and the probability density function (pdf) assumed to be:

$F(x) = 1 - (1 - x^a)^b.$	(3)

$f(x) = abx^{a-1}(1-x^a)^{b-1}.$	(4)

6.2 Burr III distribution:
The Burr distribution is a system of twelve types of distribution functions based on generating the Pearson differential equation.
The Burr III distribution proposed by Burr (1942) attracts special attention because it contains several families of non-normal distributions like gamma distribution.
The cdf and pdf of the Burr III distribution are given by:

$F(x) = (1+x^{-c})^{-k}.$	(5)
$f(x) = ckx^{-(c+1)}(1+x^{-c})^{-(k+1)}.$	(6)

6.3 Kumaraswamy-Burr III Distribution:

A composite distribution of Kumaraswamy and Burr III distributions are referred to as Kumaraswamy-Burr III distribution, which is proposed by Behairy (2016).

$F(x) = 1-(1-(1+x^{-c})^{-ak})^b.$	(7)

The cdf and pdf of the Kum-B III (a, b, c, k) distribution can be obtained as follows:
and the probability density function (pdf) assumed to be:

$f(x) = abckx^{-(c+1)}$	
$\times (1+x^{-c})^{-(ak+1)}$	(8)
$\times (1-(1+x^{-c})^{-ak})^{b-1}.$	

7. The Transmuted Kumaraswamy-Burr III Distribution:

By using the quadratic rank transmutation map which was introduced by Shaw and Buckley (2007), a random variable X is said to have The Transmuted Kumaraswamy-Burr III Distribution if its cdf is given by:

$F(x) = 1-[1-(1+x^{-c})^{-ak}]^b$	(9)
$\times [1+\lambda(1-(1+x^{-c})^{-ak})^b],$	

$where\ x > 0\ , a, b, c, k > 0\ , and\ |\lambda| \le 1.$

$f(x) = abckx^{-(c+1)}(1+x^{-c})^{-(ak+1)}$	(10)
$(1-(1+x^{-c})^{-ak})^{b-1}$	
$\times [1-\lambda+2\lambda(1-(1+x^{-c})^{-ak})^b].$	

By differentiation of F(x) we yield the pdf as follows:

8. Statistical Properties:

In this section we will discuss the statistical properties of the TKwB III distribution, including moments, quantile, random number generation, order statistics, probability, cumulative function of order statistics, and entropies.

8.1 Moments

$E(x^r) = \int_0^\infty x^r f(x)dx =$	
$\left[b\Gamma\left(1-\frac{r}{ak}\right)\left(\frac{(1-\lambda)\Gamma(b)}{\Gamma(1+b-\frac{r}{ak}} + \frac{2\lambda\Gamma(2b)}{\Gamma\left(1+2b-\frac{r}{ak}\right)}\right)\right].$	(11)

Moments are essential and necessary in any statistical analysis, especially in applications. They can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). If X has TKwB III distribution (a, b, c, k, λ), then the moments of this distribution are given by the following:

$E(x^4) = \int_0^\infty x^4 f(x)dx =$	
$\left[\frac{4b\left(\frac{(-1+\lambda)\Gamma(b)}{\Gamma\left(1+b-\frac{4}{ak}\right)} - \frac{2\lambda\Gamma(2b)}{\Gamma\left(1+2b-\frac{4}{ak}\right)}\right)\Gamma}{ak}\right.$	(15)

By using equation (11), we can get the four moments of the TKwB III distribution. They are given below:

$E(x) = \int_0^\infty xf(x)dx =$	
$\left[\frac{b\left(\frac{(-1+\lambda)\Gamma(b)}{\Gamma\left(1+b-\frac{1}{ak}\right)} - \frac{2\lambda\Gamma(2b)}{\Gamma(1+2b-\frac{1}{ak}}\right)\Gamma\left(\frac{-}{a}\right.}{ak}\right.$	(12)

$E(x^2) = \int_0^\infty x^2 f(x)dx =$	(13)

$\left[\dfrac{2b\left(\dfrac{(-1+\lambda)\Gamma(b)}{\Gamma\left(1+b-\frac{2}{ak}\right)}-\dfrac{2\lambda\Gamma(2b)}{\Gamma\left(1+2b-\frac{2}{ak}\right)}\right)\Gamma}{ak}\right.$	

$E(x^3)=\int_0^\infty x^3 f(x)dx=$	
$\left[\dfrac{3b\left(\dfrac{(-1+\lambda)\Gamma(b)}{\Gamma\left(1+b-\frac{3}{ak}\right)}-\dfrac{2\lambda\Gamma(2b)}{\Gamma\left(1+2b-\frac{3}{ak}\right)}\right)\Gamma}{ak}\right.$	(14)

8.2 The moment generation function (MGF):

The moment generation function (MGF) produces all the moments of the random variable X by differentiation. The (MGF) of the random variable X having the TKwB III distribution if it exists is given by:

$M_x(x)=E(e^{tx})=\int_0^\infty e^{tx}f(x)\,dx$	
$\left[bt^i\Gamma\left(1-\frac{i}{ak}\right)\left(\dfrac{(1-\lambda)\Gamma(b)}{\Gamma(1+b-\frac{i}{ak}}+\dfrac{2\lambda\Gamma(2b)}{\Gamma\left(1+2b-\frac{i}{ak}\right)}\right)\right].$	(16)

8.3 The quantile function:

The quantile function plays an important character in simulating random samples for the distributions. The characteristics of a distribution like the median, kurtoses, and skewness. The qth quantile x_q of the $TkwB$ III distribution is defined as:

$x_q=\left[\left\{1-\left[1-\dfrac{(1+\lambda)\mp\sqrt{\Delta}}{2\lambda}\right]^{\frac{1}{b}}\right\}^{\frac{-1}{ak}}-1\right]^{\frac{1-}{c}}.$	(17)

Where $\Delta=(1+\lambda)^2-4\lambda q=1+(2-4q)\lambda+\lambda^2$.

From this function we can obtain the first quartile, the median and the third quartile of the TKwB III distribution respectively.

8.4 Rényi Entropies:

It is well known that entropy and information can be considered as measures of uncertainty of probability distribution (Qiuping, 2008). The Rényi entropy of a random X having TKwBIII distribution is given by:

$\mathcal{J}_R(\gamma)=\frac{1}{1-\gamma}\log\left[\frac{-(1-\lambda)^{1+\gamma}+(1+\lambda)^{1+\gamma}}{2\lambda(1+\gamma)}\right].$	(18)

8.5 Order Statistics:

The appearance of order statistics is made in many areas of statistical theory and practice. Let $X_1, X_2, X_3, \ldots\ldots, X_n$ be a random sample from theTKwB III distribution. Let $X_{i:n}$ denote the ith order statistics. The pdf of $X_{i:n}$ would be shown as:

$$f_{i:n}(x)=\frac{n!}{(i-1)!\,(n-i)!}\big(F(x)\big)^{i-1}(1-F(x))^{n-0i}f(x).$$

Let $X_1, X_2, \ldots\ldots, X_n$ be independently, identically distributed order random variables from the (TKwB III) distribution and have first, last and median order possibility density; function is given as follows:

$$f_{1:n}(x)=n\big(1-F(x)\big)^{n-1}f(x).$$

$\boldsymbol{f_{1:n}(x)=n\left(1-\left[\left[1-A^{-ak}\right]^b\left[1+\lambda\left(1-A^{-ak}\right)^b\right]\right]\right)^{n-1}}$	(19)
$\times\left[abckx^{-(c+1)}A^{-(ak+1)}(1-A^{-ak})^{b-1}\right]$	
$\times\left[1-\lambda+2\lambda(1-A^{-ak})^b\right].$	

Where $A=1+x^{-c}$

And

$$f_{n:n}(x)=n\big(F(x)\big)^{n-1}f(x)$$

$f_{n:n}(x)=n\left(\left[1-\left[A^{-ak}\right]^b\left[1+\lambda\left(1-A^{-ak}\right)^b\right]\right]\right)^{n-1}$	(20)
$\times\left[abckx^{-(c+1)}A^{-(ak+1)}\left(1-A^{-ak}\right)^{b-1}\right]$	
$\times\left[1-\lambda+2\lambda(1-A^{-ak})^b\right].$	

8.6 Reliability and Hazard Rate Functions:

The reliability function S(x) (The survival function), which is the probability of an item not failing prior to sometime x, is defined by:

$S(x)=1-F(x)=$	(21)

$\left[[1-A^{-ak}]^{b}[1-\lambda+2\lambda(1-A^{-ak})^{b}]\right].$	

Therefore, the hazard function is given by:

$h(x)=f(x)/S(x).$	(22)

9. parameter Estimation:

In this section, we will employ the method of maximum likelihood to estimate the parameters of the Transmuted Kumaraswamy Burr III distribution. The maximum likelihood estimators for the unknown parameters of the (TKwB III) distribution are presented to study the properties of the estimators of the (TKwB III) distribution and their performances.

9.1 properties of the Maximum Likelihood Estimators:

The maximum likelihood estimation possesses several desirable theoretical properties that contribute to its widespread adoption in both theoretical and applied statistics. The key properties include:

9.1.1 Consistency:

Consistency is one of the fundamental properties of the maximum likelihood estimators. An estimator is said to be consistent if it converges in probability to the true value of the parameter as the sample size tends to infinity.

Suppose that $X_1, X_2, X_n, \dots..$ are independently, identically distributed according to a distribution $f(x|\Theta)$,. If $\widehat{\Theta}$ is an estimator based on the sample size, so $\widehat{\Theta}$ is written as $\widehat{\Theta}_n$ to show the independency of $\widehat{\Theta}$. A sequence of estimators is said to be consistent for an estimator Θ if it converges in probability to the true value of Θ as the sample size tends to infinity, that is, for any $\epsilon > 0$.

$$\lim_{n\to\infty} P_\Theta\left(\left|\widehat{\Theta}_n-\Theta\geq\epsilon\right|\right)=0.$$

9.1.2 Efficiency:

The maximum likelihood estimator is a cornerstone of statistical inference due to its desirable theoretical properties, particularly its asymptotic efficiency.

The maximum likelihood estimators are asymptotically most efficient. Mathematically, if there exists an alternative unbiased estimator $\widehat{\Theta}$ such that:

$\sqrt{n}(\widehat{\Theta}_n-\Theta)\xrightarrow{d}N\left(0,I^{-1}(\Theta)\right).$	(23)

Where 0 is a k-dimensional mean zero vector, $\xrightarrow{d}$ represents convergence in distribution and (Θ) is the $k\times k$ dimensional Fisher information matrix.

9.1.3 Asymptotic Normality:

The asymptotic normality is one of the most important properties of the maximum likelihood estimator.

This property plays a central role in the construction of confidence intervals and hypothesis tests in large-sample statistical inference.

Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with a probability density function $f(x,\Theta)$ where Θ is unknown parameter vector. Let $\widehat{\Theta}$ be the MLE of the Θ based on a sample of size n, then the MLE $\widehat{\Theta}_n$ satisfies the following:

$$\sqrt{n}(\widehat{\Theta}_n-\Theta_0)\xrightarrow{d}N(0,I(\Theta_0)^{-1}).$$

9.2 Maximum Likelihood Estimation of the TKwBIII Distribution:

Let $X_1, X_2, X_3, \dots\dots, X_n$ be a random sample from the TKwBIII distribution with detected values $x_1, x_2, x_3, \dots\dots, x_n$, and $\Theta=(a,b,c,k,\lambda)$ be parameter vector. Then the log likelihood function of the observed sample is given by:

$l(\Theta)=nlna+nlnb+nlnc+nlnk$ $-(c+1)\sum_{i=1}^{n}\ln(x)$ $-(ak+1)\sum_{i=1}^{n}ln(A)$ $+(b-1)\sum_{i=1}^{n}\ln(1-A^{-ak})$ $\sum_{i=1}^{n}ln\left[1+\lambda-2\lambda\left(1-(1-A^{-ak})^{b}\right)\right].$	(24)

Where $A=1+x^{-c}$.

To get the parameters estimation, say $(\hat{a},\hat{b},\hat{c},\hat{k},\hat{\lambda})$, we will find the first derivatives for equation (24) with respect to the parameters (a,b,c,k,λ) and set the total log-likelihood to zero with respect to $(\hat{a},\hat{b},\hat{c},\hat{k},\hat{\lambda})$. The partial derivatives of the function with respect to each parameter is given by:

$$\frac{\partial l(\Theta)}{\partial a} = \frac{n}{a} - k\sum_{i=1}^{n} \ln(A) + k(b-1)\sum_{i=1}^{n} \frac{A^{-ak}\ln A}{(1-A^{-ak})} + \sum_{i=1}^{n} \frac{2bk\lambda A^{-ak}(1-A^{-ak})^{b-1}\ln A}{[1+\lambda-2\lambda(1-(1-A^{-ak})^{b})]}. \quad (25)$$

$$\frac{\partial l(\Theta)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln(1-A^{-ak}) + \sum_{i=1}^{n} \frac{2\lambda[1-A^{-ak}]^{b}\ln(1-A^{-ak})}{[1+\lambda-2\lambda(1-(1-A^{-ak})^{b})]}. \quad (26)$$

$$\frac{\partial l(\Theta)}{\partial k} = \frac{n}{k} - a\sum_{i=1}^{n} \ln A + a(b-1)\sum_{i=1}^{n} \frac{A^{-ak}\ln A}{(1-A^{-ak})} + 2ab\lambda\sum_{i=1}^{n} \frac{A^{-ak}[1-A^{-ak}]^{b-1}\ln A}{[1+\lambda-2\lambda(1-(1-A^{-ak})^{b})]}. \quad (28)$$

$$\frac{\partial \ln L(\Theta)}{\partial c} = \frac{n}{c} - \sum_{i=1}^{n} lnx + (ak+1)\sum_{i=1}^{n} \frac{x^{-c}lnx}{A} - (b-1)\sum_{i=1}^{n} \frac{akx^{-c}A^{-(ak+1)}lnx}{(1-A^{-ak})} - \sum_{i=1}^{n} \frac{2abk\lambda x^{-c}A^{-(ak+1)}[1-A^{-ak}]^{b-1} \text{l}}{[1+\lambda-2\lambda(1-(1-A^{-ak})^{b})]} \quad (27)$$

$$\frac{\partial \ln L(\Theta)}{\partial \lambda} = \sum_{i=1}^{n} \frac{1-2(1-(1-(1+x^{-c})^{-ak})^{b})}{[1+\lambda-2\lambda(1-(1-(1+x^{-c})^{-ak})^{b})]}. \quad (29)$$

The maximum likelihood indicates that a, b, c , k, λ say $\hat{a}, \hat{b}, \hat{c}, \hat{k}, \hat{\lambda}$ is taken by setting the first partial derivatives to zero and solving the system of nonlinear equation simultaneously. It is much better to use nonlinear optimizatio n algorithms such as a Newton-Raphson algor ithm to get the estimate. There are many mea sures that can be used to get information abo ut the performance of the estimator as bias, s quare error, and mean square error.

9.3 Asymptotic Distribution:

The study of approximations to distributions played a major part in statistical developments in the first half of this century, and included important work by prominent scholars, such as Charlier, Edgeworth, Pearson, and so many others (Wallace, 1958).

So, since the maximum likelihood estimators of the unknown parameters cannot be obtained in a closed form expression, constructing confidence intervals for the parameters $\Theta = (a, b, c, k \; and \; \lambda)$ bcomes a challenging task. As a result, we must use the large sample approximation.

The asymptotic distribution of the MLE $\hat{\Theta} = (\hat{a}, \hat{b},, \hat{k} \; and \; \hat{\lambda})$ is given by: $(\hat{\Theta} - \Theta) \xrightarrow{d} N_4(0, I^{-1}(\Theta))$, (here $\xrightarrow{d}$ stands for convergence in distribution). The asymptotic variance-covariance matrix of MLE for the parameters $\Theta = (a, b, c, k \; and \; \lambda)$ is given by the elements of the Fisher information matrix:

$$I(\Theta) = E\left[\frac{\partial^2 l(\Theta)}{\partial\Theta_i \partial\Theta_j}\right].$$

9.4 The Simulation Study:

The technique depends on generating a large number of samples commonly referred to as simulation, and it is a fundamental tool in statistical analysis, especially when analytical solutions are difficult or impossible to obtain. In simulation, artificial data are generated according to a known or assumed model in order to study the behavior of statistical procedures under controlled conditions. In this section, a simulation study is conducted to evaluate the performance of the maximum likelihood estimators (MLEs) for the Transmuted Kumaraswamy Burr III distribution. To assess the finite-sample performance of the MLEs, the following simulation steps are performed:

1. Use the inversion method to generate two thousand samples of size n from the TKwBIII distribution, i.e., to generate values of:

$X = \left[\left\{1 - \left[1 - \frac{(1+\lambda) \mp \sqrt{\Delta}}{2\lambda}\right]^{\frac{1}{b}}\right\}^{-\frac{1}{ak}} - 1\right]^{\frac{1}{c}}.$	(30)

Where $\Delta = (1+\lambda)^2 - 4\lambda q = 1 + (2 - 4q)\lambda + \lambda^2$.

2. Set initial values for the parameters $(a_0, b_0, c_0, k_0, \lambda_0)$.
3. Choose different sample size n (say, 10, 30, 50, 80, 100, 150 and 200) to study the effect of sample size on the performance of the estimators.
4. Put the values that you obtain in step (3) in the likelihood equations (25) – (29). Then solve the likelihood equations using Newton-Raphson method.
5. The solutions that we obtain in step (4) are the maximum likelihood estimators of $a, b, c, k, and\ \lambda$ say $(\hat{a}, \hat{b}, \hat{c}, \hat{k}, \text{and } \hat{\lambda})$ for the samples obtained in step (3).
6. Then compute the statistical measures bias, mean squared errors and standard errors of the MLEs for the two thousand samples obtained in step (3). The standard errors (SE) are computed by inverting the observed information matrix. Bias and MSE are given by:

$$Bias(\widehat{\Theta}) = (\widehat{\Theta}_i - \Theta),$$

$$MSE(\widehat{\Theta}) = (\widehat{\Theta}_i - \Theta)^2,$$

for $\Theta = (a, b, c, k, \lambda)$.

And compute the average estimates, along with biases, standard errors SE $(\widehat{\Theta})$ and MSE are given by:

$Bias(\widehat{\Theta}) = \frac{\sum_{i=1}^{2000}(\widehat{\Theta}_i - \Theta)}{2000}.$	(31)

$MSE(\widehat{\Theta}) = \frac{\sum_{i=1}^{2000}(\widehat{\Theta} - \Theta)^2}{2000}.$	(32)

For $\Theta = (a, b, c, k, \lambda)$.

7. Repeat steps (3), (4), (5) and (6), 2000 times (iteration) for n= 10, 30, 50, 80, 100, 150, and 200, so computing $Bias(\widehat{\Theta})$, $MSE(\widehat{\Theta})$ and $SE(\widehat{\Theta})$ for $\Theta = a, b, c, k, \lambda$ and n= 10, 30, 50, 80, 100, 150 and 200.

where $\widehat{\Theta} = \hat{a}, \hat{b}, \hat{c}, \hat{k}, \text{and } \hat{\lambda})$ is the maximum likelihood estimators of $\Theta = a, b, c, k, \lambda$.

8. Analyse the results numerically and graphically to understand how the estimators behave under different sample size.
   The results of the simulation on the parameters of the Transmuted Kumaraswamy Burr III distribution are given in Table (1). This table includes the average estimates bias, the average mean squared errors (MSE), and the average standard errors (SE) of the maximum likelihood estimators of the parameters. This presents the results for sample sizes 10, 30, 50, 80, 100, 150 and 200.

The results of the simulation indicate that:

1. The estimates of all parameters exhibit insignificant bias across all sample sizes. The biases appear positive for parameters $\hat{a}$, $\hat{\lambda}$, and $\hat{c}$, while it is negative for parameters $\hat{k}$ and $\hat{b}$. This indicates that the estimators for $\hat{a}$, $\hat{\lambda}$, and $\hat{c}$, tend to overestimate the true parameter values, showing positive bias. In contrast, the estimators for $\hat{k}$ and $\hat{b}$ show negative biases. In general, the bias tends to decrease to zero as the values of the sample size increases.
2. Furthermore, the MSE value for most parameters tends to decrease toward zero as $n \to \infty$. These results further verify the consistency property of the estimators.
3. The standard error (SE) of the estimators decreases as the sample size increases. These results further verify the consistency property of the estimators.

10. Applications:

The statistical measures used to compare different models in order to select the one that best represents the data. These criteria assess the goodness of fit of each model based on the likelihood function. We will give the values of the maximized log-likelihood (-2ℓ), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) for the data set for the TKwBIII and its submodules. These criteria are widely used to evaluate statistical models and help identify the one that best represents the data. We also consider minimum values rule of AIC, BIC, CAIC and HQIC important to choose the best model to fit. These statistics are given by:

$$AIC = 2K - 2l(\hat{\Theta}).$$

$$CAIC = AIC + \frac{2k(k+1)}{n-k-1}.$$

$$BIC = klog(n) - 2l(\hat{\Theta}).$$

$$HQIC = 2klog\big(log(n)\big) - 2l(\hat{\Theta}).$$

We use the data set that consist of 128 bladder cancer patients given in Lee & Wang (2003). The data are given as follows:

0.08, 2.09 ,3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.74, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.69, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62 ,43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.46, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.73, 12.02, 2.02, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table (1): Estimated parameters, Bias, MSE, and SE of the TKwBIII distribution

Sample								
N		**10**	**30**	**50**	**80**	**100**	**150**	**200**
Estimate	$\hat{a}$	**1.3489377**	**1.3489376**	**1.3489376**	**1.3489377**	**1.3489378**	**1.3489378**	**1.3489377**
	$\hat{b}$	**0.8010601**	**0.8010601**	**0.8010601**	**0.8010602**	**0.8010603**	**0.8010602**	**0.80106**
	$\hat{c}$	**0.5710638**	**0.5710638**	**0.5710638**	**0.5710638**	**0.5710639**	**0.5710639**	**0.5710639**
	$\hat{k}$	**0.1756811**	**0.1756811**	**0.1756811**	**0.1756811**	**0.1756811**	**0.1756811**	**0.1756811**
	$\hat{\lambda}$	**0.5373278**	**0.5373277**	**0.5373277**	**0.5373279**	**0.5373282**	**0.5373279**	**0.5373277**
Biases	$\hat{a}$	**0.0848938**	**0.0282979**	**0.0169788**	**0.0106117**	**0.0084894**	**0.0056596**	**0.0042447**
	$\hat{b}$	**-0.069894**	**-0.023298**	**-0.013979**	**-0.008737**	**-0.006989**	**-0.00466**	**-0.003495**
	$\hat{c}$	**0.0371064**	**0.0123688**	**0.0074213**	**0.0046383**	**0.0037106**	**0.0024738**	**0.0018553**
	$\hat{k}$	**-0.082432**	**-0.027477**	**-0.016486**	**-0.010304**	**-0.008243**	**-0.005495**	**-0.004122**
	$\hat{\lambda}$	**0.0437328**	**0.0145776**	**0.0087466**	**0.0054666**	**0.0043733**	**0.0029155**	**0.0021866**
MSE	$\hat{a}$	**0.007207**	**0.0008008**	**0.0002883**	**0.0001126**	**0.0000721**	**0.000032**	**0.000018**
	$\hat{b}$	**0.0048852**	**0.0005428**	**0.0001954**	**0.0000763**	**0.0000489**	**0.0000217**	**0.0000122**
	$\hat{c}$	**0.0013769**	**0.000153**	**0.0000551**	**0.0000215**	**0.0000138**	**6.1195E-6**	**3.4422E-6**
	$\hat{k}$	**0.006795**	**0.000755**	**0.0002718**	**0.0001062**	**0.000068**	**0.0000302**	**0.000017**
	$\hat{\lambda}$	**0.0019126**	**0.0002125**	**0.0000765**	**0.0000299**	**0.0000191**	**8.5003E-6**	**4.7814E-6**
SE	$\hat{a}$	**0.1127416**	**0.0650914**	**0.0504196**	**0.0398602**	**0.035652**	**0.0291098**	**0.0252098**
	$\hat{b}$	**0.0664816**	**0.0383832**	**0.0297315**	**0.0235048**	**0.0210233**	**0.0171655**	**0.0148657**
	$\hat{c}$	**0.0319064**	**0.0184211**	**0.014269**	**0.0112806**	**0.0100897**	**0.0082382**	**0.0071345**
	$\hat{k}$	**0.0339267**	**0.0195876**	**0.0151725**	**0.0119949**	**0.0107286**	**0.0087598**	**0.0075862**
	$\hat{\lambda}$	**0.3681124**	**0.2125298**	**0.1646249**	**0.1301474**	**0.1164074**	**0.0950462**	**0.0823124**

Table (2): Descriptive Statistics

N	Mean	Median	SD	Variance	Skewness	Kurtosis	Minimum	Maximum
128	9.366	6.395	10.508	110.425	3.287	18.483	0.08	79.05

Table (3): Estimated parameters and their standard errors for second real data set

Distribution	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{k}$	$\hat{\lambda}$
TKwBIII(a,b,c,k,λ)	4.1641473 (0.2187145)	2.4016366 (0.6940636)	0.7604717 (0.0970841)	1.5012291)0.0628727(	0.001 (0.1386841)
KwBIII (a,b,c,k)	1.7437942 (0.5111928)	3.5913439 (1.8447629)	0.8629396 (0.1171909)	4.6974508 (1.3810843)	- -
TBIII (c,k,λ)	- -	- -	1.0385453 (0.0302406)	4.5075778 (0.2472286)	0.001 (0.2500305)
BIII(c,k)	- -	- -	1.0387799 (0.0605853)	4.5059546)(0.4440249	- -
TKw (a,b,λ)	0.3610442 0.0198129)(	3.8195333 (0.3793729)	- -	- -	0.7506967 (0.0955356)
Kw (a,b)	0.3225722 (0.0230834)	4.8673441 (0.6637398)	- -	- -	- -

Table (4): Log-likelihood and information criteria for real data set.

Distribution	-2ℓ	AIC	AICc	BIC	HQIC
TKwBIII(a,b,c,k,λ)	816.73112	826.73112	840.99127	827.05898	832.52509
KwBIII(a,b,c,k)	837.18527	845.18527	856.5934	845.51048	849.82045
TBIII(c,k)	833.56317	839.56317	848.11926	839.75672	843.03955
BIII(c,k)	833.55206	837.55206	843.25612	837.64806	839.86965
TKw(a,b,λ)	1100.1128	1106.1128	1114.6689	1106.3063	1109.5892
Kw(a,b)	1080.6331	1084.6331	1090.3372	1084.7291	1086.9507

11.Conclusions and Recommendations:

This section is divided into two subsections: the conclusions and the recommendations. The conclusions subsection presents a summary of the key findings and insights obtained from the analysis, while the recommendations subsection provides practical suggestions and directions for future research based on the study's results.

11.1 Conclusions

In this study, a new probability distribution called the Transmuted Kumaraswamy Burr III distribution was introduced and compared with some sup-model distributions. The study also explored several important statistical properties, such as the quantile function, moments, moment generating function, mean and median deviations, and Rényi entropy. To estimate the model parameters, the Maximum Likelihood Estimation (MLE) method was applied. A simulation study was carried out to assess the performance of the MLEs. The simulation results indicated that the estimators were approximately unbiased, and the mean squared error (MSE) values consistently decreased with increasing sample size, confirming the estimators' efficiency and consistency.

To evaluate the effectiveness of the Transmuted Kumaraswamy Burr III distribution (TKwB III), a real-data set was analysed. The findings consistently favoured the proposed distribution over its sup-models and alternative distributions. This conclusion was supported by various statistical measures, including likelihood values, and information criteria (AIC, BIC, AICc and CAIC).

11.2 Recommendations and further Research:

We now recommend some outlines for future work to extend these results. The recommendations include the following:

- In this study, we used the maximum likelihood estimation method to estimate parameters for TKwB III distribution. Although this method performed well in this study, we suggest other estimation techniques be explored, such as Bayesian estimation, method of moments, or percentile-based estimation, especially in cases where MLE may be difficult to compute or sensitive to small sample sizes.

- In this study we add a new parameter to the primary distribution. Future research could focus on extending the proposed distribution by incorporating additional parameters or constructing new families of distributions based on it, to enhance its ability to model diverse types of data.
- Future studies could improve confidence interval techniques by investigate bootstrap methods or Bayesian credible intervals for more accurate confidence interval estimation, particularly for small or moderate sample sizes.

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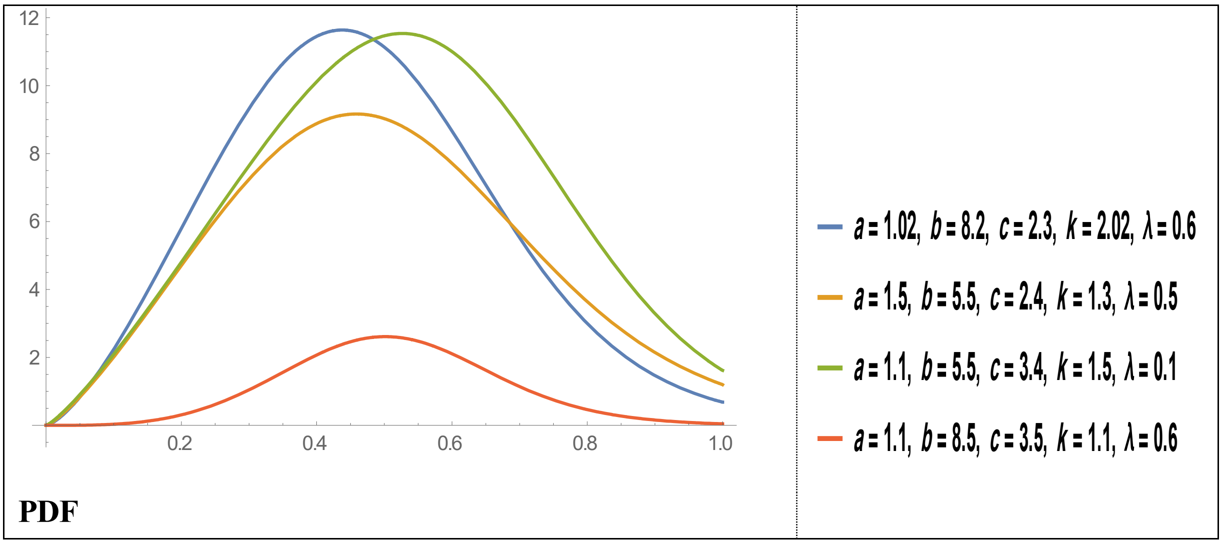

Figure (1) The PDF of the TKwBIII function distribution.

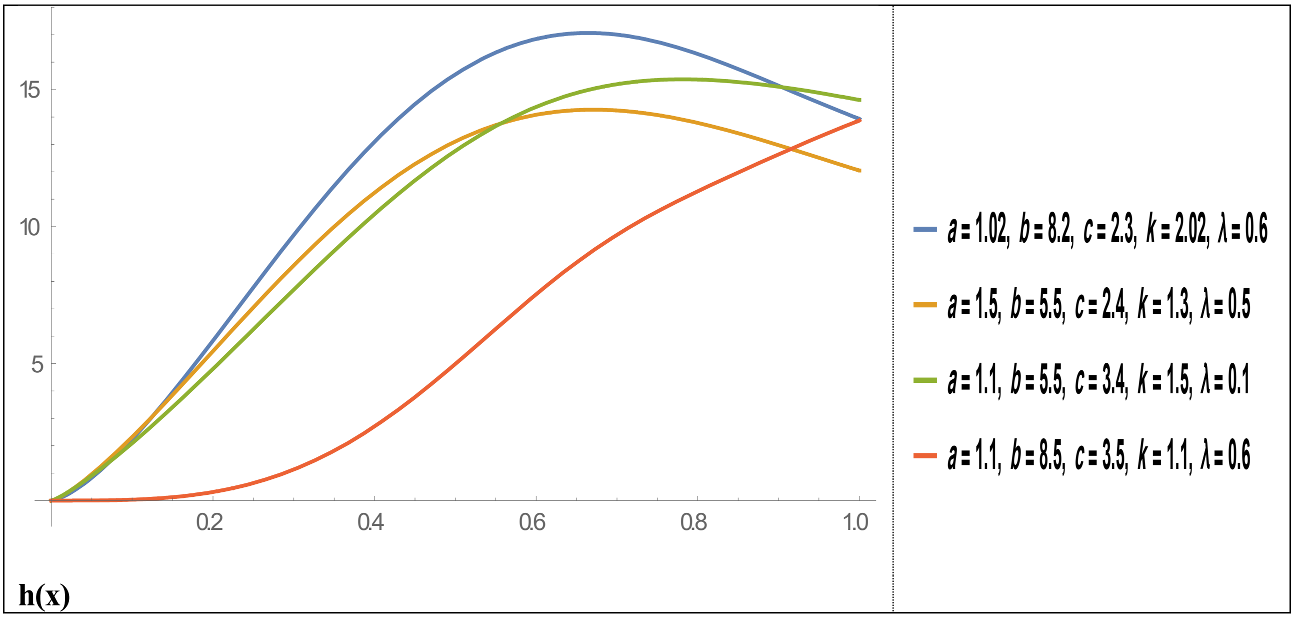

Figure (2): Hazard Function of the TKwBIII distribution at different parameter values.