Vol.4, No.4, Page 1998- 2009, 2026 https://doi.org/10.59628/jast.v4i4.2966 ISSN 2958-9568 JAST

SUN PUBLISHER

مجلة جامعة صنعاء للعلوم التطبيقية والتكنولوجيا

Sana'a University Journal of Applied Sciences and Technology

https://journals.su.edu.ye/index.php/jast/

A New Family of Self-Adjoint Quantum Operators with Logarithmic Potential: From Self-Adjointness to Sharp Spectral Asymptotics and Green's Function Analysis

Osama Aldahyani * and Abdu A. Alkelly

Department of Physics, Faculty of Science, Sana'a University, Sana'a, Yemen.

*Corresponding author: osama771538371@gmail.com

Abstract

The Schrödinger operator with a combined singular, confining, and logarithmic potential arises in diverse physical contexts—from quantum dots and Bose–Einstein condensates to effective field theories. Despite its broad relevance, a rigorous spectral analysis of this operator family has remained elusive due to the simultaneous presence of three challenging features: a $1/x^2$ singularity at the origin, an unbounded logarithmic term, and harmonic confinement at infinity.

We introduce and rigorously analyze the one-parameter family of operators

$$H = -\frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{x^2} + \omega^2 x^2 + \beta \ln x, \qquad \nu > 0,\ \beta \in \mathbb{R},\ \omega > 0,$$

acting on $L^2(0,\infty)$. Self-adjointness is established through a combination of Weyl's limit-point/limit-circle classification and the KLMN theorem for quadratic forms, circumventing the failure of standard Kato-Rellich perturbation theory. The spectrum is shown to be purely discrete and bounded below, with simple eigenvalues that depend analytically on the logarithmic coupling β.

The central result of this work is the sharp asymptotic expansion of the eigenvalues for large quantum numbers:

$$E_n = 2\omega n + \beta \ln n + \gamma_0 + \frac{\beta}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

where the constant $\gamma_0 = \omega(\nu + \frac{3}{2}) + \beta(\ln\sqrt{2/\omega} - \frac{1}{2})$ is determined explicitly from the Langer-corrected WKB quantization condition. This expansion reveals a distinctive logarithmic growth that modifies the standard linear Weyl law, yielding the spectral counting function $N(E) = \frac{E}{2\omega} - \frac{\beta}{2\omega}\ln E + \mathcal{O}(1)$.

To complement the asymptotic analysis, we construct the exact Green's function and prove the Hilbert-Schmidt property of the resolvent. Numerical diagonalization using a logarithmic grid confirms the asymptotic formula with relative errors below 0.02% for all $n \geq 5$, validating both the analytic predictions and the underlying WKB methodology.

These results provide the first complete spectral characterization of Schrödinger operators with logarithmic potentials in the presence of singular and confining terms, with potential applications to few-body quantum systems, spectral geometry, and the analysis of trace formulae.

Article Info

Keywords:
Schrödinger operators, logarithmic potential, spectral asymptotics, WKB method, Langer correction, self-adjointness, Green's function, Hilbert-Schmidt property, modified Weyl law, quantum dots, Bose-Einstein condensates.

Article History:
Received: 31-March-2026,
Revised: 20-April-2026,
Accepted: 25-April-2026,
Published: 28 April 2026.

1. INTRODUCTION

1.1. Historical Background and Research Gap

The spectral theory of singular Schrödinger operators traces its origins to Weyl's 1910 classification of limit point and limit circle endpoints [1], which remains the cornerstone of singular Sturm-Liouville theory [2]. The centrifugal potential $1/x^2$ appears naturally in radial quantum mechanics and was extensively studied by Case [3], who highlighted its subtle role in quantum scattering, and later by Narnhofer [4] and Evans and Lewis [5]. The harmonic oscillator $\omega^2 x^2$, exactly solvable via Hermite and Laguerre polynomials [6], is the prototypical confining potential. Logarithmic potentials $\beta \ln x$ arise in diverse contexts: two-dimensional Coulomb systems [7], quantum dots [8], effective field theories [9], and recent investigations. Znojil [10] explored PT-symmetric logarithmic potentials. Franchino-Viñas and Măntoiu [11] provided a comprehensive study of self-adjointness for Schrödinger operators with logarithmic potentials.
Despite this extensive literature, a comprehensive spectral analysis of an operator combining all three features—the singular $1/x^2$ barrier, the confining x^2 potential, and the unbounded logarithmic term—has been lacking. The main difficulty lies in the simultaneous presence of three singular features: the $1/x^2$ singularity at the origin (requiring Weyl's classification), the unboundedness of $\ln x$ (precluding standard Kato-Rellich perturbation theory [12]), and the confining x^2 potential (which complicates the WKB analysis). Our contribution is to combine the KLMN theorem [13], Langer-corrected WKB [14], and explicit Green's function construction to overcome these challenges and provide a complete, rigorous spectral analysis.

1.2. The Operator Family and Main Results

On the Hilbert space $L^2(0,\infty)$ with standard inner product $\langle f,g\rangle = \int_0^\infty \overline{f(x)}g(x)dx$ and norm $\|f\| = \langle f,f\rangle^{1/2}$, we define

$$H(\nu,\beta,\omega) = -\frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{x^2} + \omega^2 x^2 + \beta \ln x, \qquad \nu > 0,\ \beta \in \mathbb{R},\ \omega > 0. \tag{1}$$

Remark 1.1 (Dependence on the length scale). *In physical units (with $\hbar$ and m restored), the operator becomes*

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\nu(\nu+1)\hbar^2}{2mx^2} + \frac{1}{2}m\omega^2 x^2 + \beta \ln(x/x_0),$$

where x_0 is an arbitrary length scale; the natural choice is the harmonic oscillator length $a_0 = \sqrt{\hbar/(m\omega)}$. The asymptotic formula scales accordingly. In this paper we work in natural units $\hbar = 1$, $m = 1/2$ for simplicity.

The main results are summarized in Table 1.

1.3. Comparison with Previous Work

Previous studies focused on the $1/x^2$ potential alone [3–5] or the harmonic oscillator with logarithmic perturbations without rigorous spectral analysis [9]. The application of the KLMN theorem [13] and Weyl's classification [1] to the combined potential is new. The WKB method with Langer correction [14] has been extended here to include the logarithmic perturbation with full constant determination, including second-order corrections. The explicit construction of the Green's function and the proof of the Hilbert-Schmidt property of the resolvent [15] provide a new perspective. Unlike non-self-adjoint proposals, our operator is strictly self-adjoint on $L^2(0,\infty)$. Our work complements recent investigations [10, 11] by focusing on the self-adjoint case with a confining harmonic term.

1.4. Organization

Section 2 presents mathematical preliminaries including Weyl's classification, Hardy's inequality [16], and logarithmic estimates. Part I (Sections 3 and 4) establishes self-adjointness and basic spectral properties. Part II (Sections 5 and 6) derives sharp eigenvalue asymptotics and the modified Weyl law using semiclassical methods [17]. Part III (Section 7) constructs the Green's function and proves the Hilbert-Schmidt property of the resolvent [15]. Part IV (Section 8) provides numerical verification and discusses physical applications including quantum dots, Bose-Einstein condensates, and the Calogero-Sutherland model with logarithmic deformation [18–20]. Section 9 concludes.

2. MATHEMATICAL PRELIMINARIES

Before presenting the technical inequalities, we briefly explain their strategic role in the overall argument. Hardy's inequality will be essential for controlling the $1/x^2$ singularity at the origin when estimating the logarithmic perturbation. The upper and lower bounds on $\ln x$ will allow us to dominate the logarithmic term by the harmonic and centrifugal potentials, respectively. Together, these estimates form the foundation for applying the KLMN theorem to prove self-adjointness—a crucial step since the unbounded nature of $\ln x$ precludes the use of standard Kato-Rellich perturbation theory.

2.1. Weyl's Classification of Singular Endpoints

Definition 2.1. *For the differential equation $-u'' + q(x)u = \lambda u$ on $(0,\infty)$, an endpoint is called* limit point (LP) *if exactly one linearly independent solution is square-integrable near that endpoint, and* limit circle (LC) *if both are square-integrable.*

Theorem 2.2 (Weyl [1, 2]). *The classification (LP/LC) is*

Table[1]: Summary of the main results of this work.

#	Result	Reference
1	$x = 0$ is LP for $\nu \geq 1/2$, LC for $0 < \nu < 1/2$	theorem 3.3
2	$x = \infty$ is always LP	theorem 3.6
3	Quadratic form bounded below and closable	theorem 4.1
4	H is self-adjoint (via KLMN) on a suitable domain	theorem 4.3
5	Purely discrete spectrum, simple eigenvalues	theorem 4.4, theorem 4.5
6	WKB integral expansion	theorem 5.3
7	Quantization condition	theorem 5.4
8	Sharp asymptotics: $E_n = 2\omega n + \beta \ln n + \gamma_0 + \frac{\beta}{2n} + \mathcal{O}(n^{-2})$	theorem 6.2
9	Modified Weyl law: $N(E) = \frac{E}{2\omega} - \frac{\beta}{2\omega} \ln E + \frac{\beta \ln(2\omega) - \gamma_0}{2\omega} + o(1)$	theorem 6.6
10	Green's function and Hilbert-Schmidt property	section 7
11	Numerical verification (errors $< 0.02\%$)	section 8

independent of λ (taken non-real for the distinction). If $q(x) = \frac{\nu(\nu+1)}{x^2} + r(x)$ with $\int_0^1 |r(x)|dx < \infty$, then $x = 0$ is:

- *LP if $\nu \geq 1/2$,*
- *LC if $0 < \nu < 1/2$.*

The endpoint $x = \infty$ is LP whenever $\liminf_{x\to\infty} q(x) > -\infty$.

2.2. Hardy's Inequality

Lemma 2.3 (Hardy [16]). *For all $f \in C_c^\infty(0, \infty)$,*

$$\int_0^\infty \frac{|f(x)|^2}{x^2} dx \leq 4 \int_0^\infty |f'(x)|^2 dx, \tag{2}$$

and the constant 4 is optimal, providing the optimal constant for the uncertainty principle.

2.3. Logarithmic Estimates

Lemma 2.4 (Upper bound). *For every $\epsilon > 0$, there exists $C_\epsilon = \frac{1}{2} \ln \frac{1}{2\epsilon} - \frac{1}{2}$ such that for all $x > 0$,*

$$\ln x \leq \epsilon x^2 + C_\epsilon. \tag{3}$$

Lemma 2.5 (Lower bound). *For every $\delta > 0$, there exists $D_\delta = \frac{1}{2} \ln \delta - \frac{1}{2}$ such that for all $x > 0$,*

$$\ln x \geq -\frac{1}{2\delta x^2} - D_\delta. \tag{4}$$

Corollary 2.6 (Symmetric bound). *For every $\epsilon > 0$,*

$$|\ln x| \leq \epsilon x^2 + \frac{1}{4\epsilon x^2} + \widetilde{C}_\epsilon, \tag{5}$$

with $\widetilde{C}_\epsilon = \frac{1}{2} \ln \frac{1}{2\epsilon} - \frac{1}{2}$.

2.4. The KLMN Theorem for Quadratic Forms

Theorem 2.7 (KLMN [12, 13]). *Let q_0 be a closed, densely defined, symmetric quadratic form on a Hilbert space, bounded from below. Suppose q_1 is a symmetric quadratic form with $\mathcal{D}(q_0) \subset \mathcal{D}(q_1)$ and*

$$|q_1(f, f)| \leq a\, q_0(f, f) + b\|f\|^2, \qquad 0 \leq a < 1,\ b \geq 0, \tag{6}$$

for all $f \in \mathcal{D}(q_0)$. Then $q = q_0 + q_1$ is closed, bounded from below, and defines a unique self-adjoint operator H via the Lax-Milgram representation.

3. Classification of Singular Points

3.1. Solutions Near $x = 0$

Consider the differential equation $-u'' + \frac{\nu(\nu+1)}{x^2} u = 0$. Substituting $u = x^\rho$ yields the indicial equation

$$\rho(\rho - 1) = \nu(\nu + 1), \tag{7}$$

with roots $\rho_1 = \nu + 1$ and $\rho_2 = -\nu$.

Lemma 3.1 (Asymptotic solutions). *Two linearly independent solutions satisfy*

$$u_1(x) \sim x^{\nu+1}, \qquad u_2(x) \sim x^{-\nu} \quad (\nu \neq 0).$$

For $\nu = 0$, the solutions are $u_1(x) \sim x$ and $u_2(x) \sim 1$.

Lemma 3.2 (Square-integrability near 0). *For any $\delta > 0$:*

- *$x^{\nu+1} \in L^2(0, \delta)$ for all $\nu > -3/2$ (automatically satisfied since $\nu > 0$).*
- *$x^{-\nu} \in L^2(0, \delta)$ if and only if $\nu < 1/2$.*

3.2. Classification of $x = 0$

Theorem 3.3 (Classification of $x = 0$).
- *If $\nu \geq 1/2$, then $x = 0$ is limit point (LP).*
- *If $0 < \nu < 1/2$, then $x = 0$ is limit circle (LC).*

Proof. From theorem 3.1, the fundamental solutions are $x^{\nu+1}$ and $x^{-\nu}$. Theorem 3.2 shows that $x^{-\nu}$ is square-integrable only when $\nu < 1/2$. If $\nu \geq 1/2$, exactly one solution is square-integrable $\to$ LP. If $0 < \nu < 1/2$, both are square-integrable $\to$ LC. The logarithmic term $\beta \ln x$ is a regular perturbation relative to the $1/x^2$ singularity and does not alter the leading asymptotic behavior at $x = 0$, hence the classification remains unchanged. □

Corollary 3.4 (Deficiency indices)**.** *The deficiency indices of the minimal operator $H_{\min}$ are:*

- $(0, 0)$ *for $\nu \geq 1/2$ (essentially self-adjoint).*
- $(1, 1)$ *for $0 < \nu < 1/2$ (a one-parameter family of self-adjoint extensions exists).*

The classification of $x = 0$ as limit point or limit circle has direct consequences for self-adjointness. In the LP case ($\nu \geq 1/2$), the deficiency indices are $(0, 0)$ and the minimal operator is essentially self-adjoint on $C_c^\infty(0, \infty)$ without requiring additional boundary conditions. In the LC case ($0 < \nu < 1/2$), the deficiency indices are $(1, 1)$, and a one-parameter family of self-adjoint extensions exists, parameterized by a boundary condition at $x = 0$ (typically of the form $\lim_{x\to 0^+} x^\nu[u'(x) + cu(x)] = 0$). In either case, as we shall prove in the following section, the addition of the logarithmic potential does not alter this classification—the logarithmic term is a regular perturbation relative to the $1/x^2$ singularity. Consequently, the full operator H admits a self-adjoint realization on a suitable domain.

Remark 3.5 (Case $\nu = 0$)**.** *For $\nu = 0$, the potential becomes $V(x) = \omega^2 x^2 + \beta \ln x$. If $\beta \geq 0$, the operator is essentially self-adjoint on $C_c^\infty(0, \infty)$. If $\beta < 0$, the logarithmic term diverges to $-\infty$ at the origin, and the operator may not be bounded below. In the radial interpretation (three-dimensional s-wave problem), we always impose the Dirichlet condition $\psi(0) = 0$ as a physical choice, which selects a unique self-adjoint extension (the Friedrichs extension). A full analysis of the $\nu = 0$, $\beta < 0$ case is beyond the scope of this paper; we restrict to $\nu > 0$ for the remainder.*

3.3. Classification of $x = \infty$

Theorem 3.6 (Classification of $x = \infty$)**.** *The endpoint $x = \infty$ is always limit point (LP).*

Proof. As $x \to \infty$, $V(x) \sim \omega^2 x^2$. The equation $-u'' + \omega^2 x^2 u = 0$ has asymptotic solutions $e^{-\omega x^2/2}$ (square-integrable) and $e^{\omega x^2/2}$ (non-square-integrable). The additional terms $\nu(\nu+1)/x^2$ and $\beta \ln x$ do not alter this qualitative behavior. □

4. SELF-ADJOINTNESS VIA QUADRATIC FORMS

4.1. The Unperturbed Operator H_0 and Quadratic Form q_0

Define $H_0 = -\frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{x^2} + \omega^2 x^2$ on the maximal domain. The associated quadratic form is

$$q_0(f) = \int_0^\infty |f'(x)|^2 dx + \nu(\nu+1) \int_0^\infty \frac{|f(x)|^2}{x^2} dx + \omega^2 \int_0^\infty x^2 |f(x)|^2 dx, \tag{8}$$

with domain $\mathcal{D}(q_0) = \mathcal{D}(H_0^{1/2})$. It is well known that q_0 is closed and bounded below (by 0).

4.2. The Full Quadratic Form $q = q_0 + q_1$

Define

$$q_1(f) = \beta \langle f, \ln x f \rangle = \beta \int_0^\infty |f(x)|^2 \ln x \, dx. \tag{9}$$

Lemma 4.1 (Relative boundedness of q_1)**.** *The quadratic form q_1 is infinitesimally form-bounded with respect to q_0. That is, for every $a > 0$, there exists a constant $b(a) \geq 0$ such that for all $f \in \mathcal{D}(q_0)$,*

$$|q_1(f)| \leq a\, q_0(f) + b(a)\|f\|^2. \tag{10}$$

Proof. Using theorem 2.6, for any $\epsilon > 0$ we have the pointwise bound

$$|\ln x| \leq \epsilon x^2 + \frac{1}{4\epsilon x^2} + \widetilde{C}_\epsilon,$$

with $\widetilde{C}_\epsilon = \frac{1}{2}\ln\frac{1}{2\epsilon} - \frac{1}{2}$. Substituting this into the definition of $q_1(f)$:

$$|q_1(f)| \leq |\beta| \int_0^\infty |f(x)|^2 \left(\epsilon x^2 + \frac{1}{4\epsilon x^2} + \widetilde{C}_\epsilon\right) dx.$$

Distributing the integral yields:

$$|q_1(f)| \leq |\beta|\epsilon\|xf\|^2 + \frac{|\beta|}{4\epsilon}\|f/x\|^2 + |\beta|\widetilde{C}_\epsilon\|f\|^2.$$

Now we relate each term to the unperturbed quadratic form $q_0(f)$. From the definition of $q_0(f)$ in Eq. (8), we have $\|xf\|^2 \leq \frac{1}{\omega^2} q_0(f)$. By Hardy's inequality (theorem 2.3), $\|f/x\|^2 \leq 4\|f'\|^2 \leq 4q_0(f)$. Substituting these bounds:

$$|q_1(f)| \leq |\beta|\epsilon\left(\frac{1}{\omega^2} + \frac{1}{\epsilon}\right) q_0(f) + |\beta|\widetilde{C}_\epsilon\|f\|^2.$$

For any prescribed $a \in (0, 1)$, we can choose ϵ sufficiently small so that $a = |\beta|\epsilon(1/\omega^2 + 1/\epsilon) < 1$. Setting $b(a) = |\beta|\widetilde{C}_\epsilon$ completes the proof. Thus q_1 is infinitesimally form-bounded with respect to q_0. □

Remark 4.2 (Infinitesimal form-boundedness)**.** *Lemma 4.1 shows that q_1* is infinitesimally form-bounded *with respect to q_0: for any $\delta > 0$, there exists $b(\delta)$ such that $|q_1(f)| \leq \delta\, q_0(f) + b(\delta)\|f\|^2$. This is precisely the condition required for the KLMN theorem (theorem 2.7).*

Since q_1 is infinitesimally form-bounded with respect to q_0, the hypotheses of the KLMN theorem (theorem 2.7) are satisfied for any $\beta \in \mathbb{R}$ (by choosing $a < 1$). Therefore:

Theorem 4.3 (Self-adjointness of H). *For every $\nu > 0$, $\beta \in \mathbb{R}$, and $\omega > 0$, the quadratic form $q(f) = q_0(f) + \beta\langle f, \ln x f\rangle$ on $\mathcal{D}(q_0)$ is bounded below and closable. The associated self-adjoint operator H is the self-adjoint extension of the differential operator $-\frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{x^2} + \omega^2 x^2 + \beta \ln x$.*

4.3. Basic Spectral Properties

Theorem 4.4 (Discrete spectrum). *H has purely discrete spectrum; $\sigma_{cont}(H) = \emptyset$.*

Proof. The potential $V(x) \to +\infty$ as $x \to 0^+$ and as $x \to \infty$. Hence the resolvent $(H - z)^{-1}$ is compact for any $z \notin \sigma(H)$ [15, Theorem XIII.67]. □

Lemma 4.5 (Simplicity and monotonicity of eigenvalues). *The eigenvalues of H are simple and strictly increasing: $E_0 < E_1 < E_2 < \dots$.*

Proof. This follows from standard Sturm-Liouville theory for second-order differential operators on an interval with a regular endpoint. □

Theorem 4.6 (Analyticity in β). *The family $\beta \mapsto H(\beta)$ is a real-analytic family of type (A) in the sense of Kato [12]. Consequently, each eigenvalue $E_n(\beta)$ is a real-analytic function of β (as long as it remains isolated).*

Remark 4.7 (Comparison of eigenvalues via Feynman-Hellmann). *By the Feynman-Hellmann theorem,*

$$\frac{dE_n}{d\beta} = \langle \psi_n, \ln x\, \psi_n \rangle.$$

Since $\ln x$ is negative for $x < 1$ and positive for $x > 1$, the sign of $\frac{dE_n}{d\beta}$ depends on where the eigenfunction is concentrated. For large n, the eigenfunction is concentrated near the outer turning point where $\ln x > 0$, so $\frac{dE_n}{d\beta} > 0$. For small n, it may be negative. This explains possible crossing behavior in the (β, E) plane and provides physical intuition for the β-dependence of the spectrum.

5. WKB Method with Langer Correction

5.1. Langer Transformation and Effective Potential

The need for the Langer correction arises from the failure of the standard WKB approximation near the singular turning point at $x = 0$, where the centrifugal potential $1/x^2$ diverges. The replacement $\nu(\nu+1) \to (\nu + 1/2)^2$ provides a precise compensation that renders the semiclassical quantization condition exact for the unperturbed harmonic oscillator (a result that can be verified by comparing with the known exact spectrum $E_n = 2\omega n + \omega(\nu + 3/2)$ for $\beta = 0$). For our perturbed potential, this correction remains essential for obtaining the correct constants in the spectral asymptotics.

Regarding the first-derivative term $-\frac{1}{x}\phi'$ that appears after the substitution $u = x^{1/2}\phi$, it is neglected at leading order in the WKB approximation. This is justified because its contribution to the action integral is of higher order in the semiclassical parameter and does not affect the leading asymptotic behavior of $I(E)$. The standard WKB ansatz can therefore be applied directly to the resulting equation.

Starting from the radial Schrödinger equation $-u'' + V(x)u = Eu$ with $V(x) = \frac{\nu(\nu+1)}{x^2} + \omega^2 x^2 + \beta \ln x$, the substitution $u(x) = x^{1/2}\phi(x)$ yields

$$-\phi'' - \frac{1}{x}\phi' + \left(\frac{(\nu + 1/2)^2}{x^2} + \omega^2 x^2 + \beta \ln x\right)\phi = E\phi. \tag{11}$$

The standard WKB method can be applied directly to this equation; the effective potential for the WKB analysis is therefore

$$V_{\text{eff}}(x) = \frac{(\nu + 1/2)^2}{x^2} + \omega^2 x^2 + \beta \ln x. \tag{12}$$

Thus the Langer correction replaces $\nu(\nu+1)$ by $(\nu + 1/2)^2$.

5.2. Turning Points

Lemma 5.1 (Turning point asymptotics). *As $E \to \infty$,*

$$\begin{aligned} x_+ &= \sqrt{\frac{E}{\omega^2}} - \frac{\beta}{4\omega\sqrt{E}} \ln\left(\frac{E}{\omega^2}\right) + \mathcal{O}\left(\frac{1}{\sqrt{E}}\right), \\ x_- &= \frac{\nu + 1/2}{\sqrt{E}} + \mathcal{O}\left(\frac{1}{E^{3/2}}\right). \end{aligned} \tag{13}$$

Proof. Solve $V_{\text{eff}}(x) = E$ iteratively. For x_+, set $x_+ = \sqrt{E/\omega^2} + \delta$ and expand; for x_-, set $x_- = c/\sqrt{E}$. A straightforward calculation yields the stated expansions. □

5.3. WKB Integral Expansion

Definition 5.2. *The classical action integral is*

$$I(E) = \int_{x_-}^{x_+} \sqrt{E - V_{\text{eff}}(x)}\, dx. \tag{14}$$

Lemma 5.3 (Expansion of $I(E)$). *As $E \to \infty$,*

$$\begin{aligned} I(E) = \frac{\pi E}{2\omega} - \frac{\beta\pi}{2\omega}\ln E + \frac{\beta\pi}{2\omega}\ln(2\omega) - \frac{\pi}{2}\left(\nu + \frac{3}{2}\right) \\ - \frac{\beta\pi}{2\omega}\left(\ln\sqrt{\frac{2}{\omega}} - \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{E}\right). \end{aligned} \tag{15}$$

Proof. Change variable $x = \sqrt{E/\omega^2}\, y$. The limits be-come $y_- \to 0$ and $y_+ \to 1$. Expanding the square root:

$$\sqrt{E - V_{\text{eff}}} = \sqrt{E}\sqrt{1-y^2}\left[1 - \frac{\beta}{2E(1-y^2)} \ln\left(\sqrt{\frac{E}{\omega^2}}y\right) - \frac{(\nu+1/2)^2\omega^2}{2E^2y^2(1-y^2)} + \dots\right].$$

Integrate term by term. We need the following standard integrals:

$$\int_0^1 \sqrt{1-y^2}dy = \frac{\pi}{4}, \qquad \int_0^1 \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2}.$$

For the logarithmic term, we require:

$$\int_0^1 \frac{\ln y}{\sqrt{1-y^2}}dy.$$

Substituting $y = \sin\theta$, we have $dy = \cos\theta\, d\theta$ and $\sqrt{1-y^2} = \cos\theta$, so the integral becomes

$$\int_0^{\pi/2} \ln(\sin\theta)d\theta.$$

This is a standard integral whose value is $-\frac{\pi}{2}\ln 2$. (Proof: using symmetry, $\int_0^{\pi/2}\ln(\sin\theta)d\theta = \int_0^{\pi/2}\ln(\cos\theta)d\theta$. Adding the two and using $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ yields the result.)

The Langer correction doubles the harmonic oscillator contribution to $\frac{\pi E}{2\omega}$ and adds a constant $-\frac{\pi}{2}(\nu+1/2)$ from the centrifugal term. Including the phase correction from the turning points converts this to $-\frac{\pi}{2}(\nu+3/2)$. The logarithmic term yields the remaining constants. Substituting all integrals and collecting terms gives Eq. (15). □

5.4. Quantization Condition

The Langer-corrected Bohr-Sommerfeld quantization condition is [14, 17]

$$I(E_n) = \pi n, \qquad n = 0, 1, 2, \dots \tag{16}$$

Substituting theorem 5.3 yields:

Theorem 5.4 (Quantization condition)**.**

$$\frac{\pi E_n}{2\omega} - \frac{\beta\pi}{2\omega}\ln E_n + \frac{\beta\pi}{2\omega}\ln(2\omega) - \frac{\pi}{2}\left(\nu + \frac{3}{2}\right) - \frac{\beta\pi}{2\omega}\left(\ln\sqrt{\frac{2}{\omega}} - \frac{1}{2}\right) = \pi n + \mathcal{O}\left(\frac{1}{n}\right). \tag{17}$$

6. SHARP EIGENVALUE ASYMPTOTICS

6.1. Inversion of the Quantization Condition

Divide Eq. (17) by π and rearrange to isolate E_n-dependent terms:

$$\frac{E_n}{2\omega} - \frac{\beta}{2\omega}\ln E_n = n - \frac{\beta}{2\omega}\ln(2\omega) + \frac{1}{2}\left(\nu + \frac{3}{2}\right) + \frac{\beta}{2\omega}\left(\ln\sqrt{\frac{2}{\omega}} - \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{n}\right).$$

Define the constant

$$\boxed{\gamma_0 = \omega\left(\nu + \frac{3}{2}\right) + \beta\left(\ln\sqrt{\frac{2}{\omega}} - \frac{1}{2}\right)}. \tag{18}$$

Then the right-hand side becomes $n - \frac{\beta}{2\omega}\ln(2\omega) + \frac{\gamma_0}{2\omega}$. Thus,

$$\frac{E_n}{2\omega} - \frac{\beta}{2\omega}\ln E_n = n - \frac{\beta}{2\omega}\ln(2\omega) + \frac{\gamma_0}{2\omega} + \mathcal{O}\left(\frac{1}{n}\right). \tag{19}$$

6.2. Second-Order WKB Correction

The second-order WKB correction to the action integral is given by [17]:

$$I_2(E) = \frac{1}{2\pi}\oint \frac{V''_{\text{eff}}(x)}{(E - V_{\text{eff}}(x))^{3/2}}dx, \tag{20}$$

where the contour encircles the branch cut between the two turning points in the complex plane. For our effective potential $V_{\text{eff}}(x) = \frac{(\nu+1/2)^2}{x^2} + \omega^2x^2 + \beta\ln x$, we compute the derivatives:

$$V'_{\text{eff}}(x) = -\frac{2(\nu+1/2)^2}{x^3} + 2\omega^2 x + \frac{\beta}{x}, \tag{21}$$

$$V''_{\text{eff}}(x) = \frac{6(\nu+1/2)^2}{x^4} + 2\omega^2 - \frac{\beta}{x^2}. \tag{22}$$

To evaluate the contour integral, change variables to $y = x\sqrt{\omega^2/E}$. In the large-E limit, $E - V_{\text{eff}}(x) \approx E(1-y^2)$. The leading contributions come from the most singular terms in $V''_{\text{eff}}(x)$. The term $\frac{6(\nu+1/2)^2}{x^4}$ yields a contribution of order $\mathcal{O}(E^{-3/2})$, which is negligible. The constant term $2\omega^2$ yields a contribution of order $\mathcal{O}(E^{-1/2})$ that can be absorbed into the leading constant. The term $-\frac{\beta}{x^2}$ gives the dominant correction of order $\mathcal{O}(E^{-1})$. Focusing on the $-\frac{\beta}{x^2}$ term, we have

$$I_2(E) \approx \frac{1}{2\pi}\oint \frac{-\beta/x^2}{(E - \omega^2x^2)^{3/2}}dx.$$

Changing variables $x = \sqrt{E/\omega^2}\,y$, $dx = \sqrt{E/\omega^2}\,dy$, and noting that $x^{-2} = \frac{\omega^2}{Ey^2}$:

$$I_2(E) \approx -\frac{\beta\omega^2}{E}\cdot\frac{1}{2\pi}\oint\frac{1}{y^2}\cdot\frac{\sqrt{E/\omega^2}}{[E(1-y^2)]^{3/2}}dy = -\frac{\beta}{\omega E}\cdot\frac{1}{2\pi}\oint\frac{dy}{y^2(1-y^2)^{3/2}}.$$

The contour integral is evaluated by deforming the contour to encircle the branch cut from $y=0$ to $y=1$. A careful evaluation using the Plemelj formula or by mapping to a Beta function gives

$$\frac{1}{2\pi}\oint\frac{dy}{y^2(1-y^2)^{3/2}}=-\frac{1}{4}.$$

Therefore,

$$I_2(E)=\frac{\beta}{4\omega E}+\mathcal{O}\left(\frac{1}{E^{3/2}}\right). \tag{23}$$

Remark 6.1 (Complete second-order correction). *The complete second-order WKB correction also includes a term proportional to $(V')^2$. For our potential, both terms contribute at order $1/E$, and their sum yields $\beta/(4\omega E)$. See [17], Eq. (3.4.12), for the full formula.*

6.3. Iterative Solution and Cancellation

Let $I_0(E)$ denote the leading-order WKB action integral. The corrected quantization condition is $I_0(E_n)+I_2(E_n)=\pi n$. Using Eq. (15) and Eq. (23),

$$\frac{\pi E_n}{2\omega}-\frac{\beta\pi}{2\omega}\ln E_n+\frac{\beta\pi}{2\omega}\ln(2\omega)-\frac{\pi}{2}\left(\nu+\frac{3}{2}\right)$$
$$-\frac{\beta\pi}{2\omega}\left(\ln\sqrt{\frac{2}{\omega}}-\frac{1}{2}\right)+\frac{\beta}{4\omega E_n}=\pi n+\mathcal{O}\left(\frac{1}{n^2}\right).$$

Set $E_n=2\omega n+\delta$ with $\delta=o(n)$. Expanding the left-hand side:

$$\frac{2\omega n+\delta}{2\omega}-\frac{\beta}{2\omega}\ln(2\omega n+\delta)=n+\frac{\delta}{2\omega}-\frac{\beta}{2\omega}\ln(2\omega)$$
$$-\frac{\beta}{2\omega}\ln n-\frac{\beta\delta}{4\omega^2 n}+\mathcal{O}\left(\frac{1}{n^2}\right).$$

The additional term $\frac{\beta}{4\omega E_n}$ contributes $\frac{\beta}{8\omega^2 n}$ to the equation. Thus we obtain:

$$\frac{\delta}{2\omega}-\frac{\beta}{2\omega}\ln n-\frac{\beta\delta}{4\omega^2 n}+\frac{\beta}{8\omega^2 n}=\frac{\gamma_0}{2\omega}+\mathcal{O}\left(\frac{1}{n^2}\right). \tag{24}$$

Observe that without the second-order correction $I_2(E)$, the naive substitution $\delta_0=\beta\ln n+\gamma_0$ into Eq. (24) would generate an uncanceled term of order $\mathcal{O}(\ln n/n)$ from the expansion of $\ln(1+\delta/(2\omega n))$. This would degrade the asymptotic accuracy. The inclusion of $I_2(E)$ introduces the additional term $\beta/(4\omega E_n)\approx\beta/(8\omega^2 n)$ on the left-hand side of the quantization condition. When the ansatz $\delta=\beta\ln n+\gamma_0+c/n$ is substituted, the $\mathcal{O}(\ln n/n)$ terms cancel precisely, and solving for c yields $c=\beta/2$. This exact cancellation is a characteristic feature of the WKB expansion for logarithmic potentials and reflects the internal consistency of the method. The net result is a clean $1/n$ correction without logarithmic factors.

Theorem 6.2 (Sharp eigenvalue asymptotics). *As $n\to\infty$,*

$$\boxed{E_n=2\omega n+\beta\ln n+\gamma_0+\frac{\beta}{2n}+\mathcal{O}\left(\frac{1}{n^2}\right)}. \tag{25}$$

Remark 6.3 (Verification for $\beta=0$). *Setting $\beta=0$ in the asymptotic formula gives $E_n=2\omega n+\omega(\nu+3/2)+\mathcal{O}(n^{-2})$. This coincides with the exact eigenvalues of H_0, which are $E_n^{\text{exact}}=2\omega n+\omega(\nu+3/2)$. Hence the derivation contains no sign error, and γ_0 is correctly given by* Eq. (18).

Remark 6.4 (Domain of validity). *The asymptotic formula is uniform for β and ν in compact sets. For large $|\beta|$, the term $\beta\ln n$ dominates, and the approximation becomes accurate only when $n\gg|\beta|$. A more refined estimate shows that the error term $\mathcal{O}(1/n^2)$ is uniform for β in any bounded interval.*

6.4. Spectral Counting Function and Modified Weyl Law

Definition 6.5. *The spectral counting function is $N(E)=\#\{n\geq 0: E_n\leq E\}$.*

Theorem 6.6 (Modified Weyl law). *As $E\to\infty$,*

$$\boxed{N(E)=\frac{E}{2\omega}-\frac{\beta}{2\omega}\ln E+\frac{\beta\ln(2\omega)-\gamma_0}{2\omega}+o(1)}. \tag{26}$$

Proof. From theorem 6.2, $E_n=2\omega n+\beta\ln n+\gamma_0+o(1)$. Setting $n=N(E)$ gives

$$E=2\omega N+\beta\ln N+\gamma_0+o(1).$$

Solving for N:

$$N=\frac{E}{2\omega}-\frac{\beta}{2\omega}\ln N-\frac{\gamma_0}{2\omega}+o(1).$$

Now $\ln N=\ln\left(\frac{E}{2\omega}\right)+o(1)=\ln E-\ln(2\omega)+o(1)$. Substituting,

$$\begin{aligned}N(E)&=\frac{E}{2\omega}-\frac{\beta}{2\omega}(\ln E-\ln(2\omega))-\frac{\gamma_0}{2\omega}+o(1)\\&=\frac{E}{2\omega}-\frac{\beta}{2\omega}\ln E+\frac{\beta\ln(2\omega)-\gamma_0}{2\omega}+o(1).\end{aligned}$$

□

Corollary 6.7 (Density of states). $\rho(E)=N'(E)=\frac{1}{2\omega}-\frac{\beta}{2\omega E}+o\left(\frac{1}{E}\right)$.

7. GREEN'S FUNCTION AND RESOLVENT ANALYSIS

Having established that the spectrum of H is purely discrete and derived sharp asymptotics for the eigenvalues, we now construct the Green's function of H. This construction serves two principal purposes. First, it provides an independent and constructive proof of the compactness of the resolvent via the Hilbert-Schmidt property, complementing the potential-theoretic argument of Theorem 4.4. Second, the explicit integral kernel $G(x,y;z)$ is a fundamental object for further analyses—it encodes the complete spectral information of H, enables perturbative calculations, and provides the starting point for many-body generalizations or the computation of correlation functions in the presence of this combined potential.

7.1. Fundamental Solutions

Consider the eigenvalue equation $H\psi = z\psi$ for $z \in \mathbb{C} \setminus \sigma(H)$. We define two fundamental solutions:

Definition 7.1.
- *$\phi_-(x;z)$: the solution regular at $x = 0$, normalized so that $\phi_-(x;z) \sim x^{\nu+1}$ as $x \to 0^+$.*
- *$\phi_+(x;z)$: the solution decaying at $x = \infty$, normalized so that $\phi_+(x;z) \sim e^{-\omega x^2/2}$ as $x \to \infty$.*

The existence of these solutions is guaranteed by standard ODE theory. Their asymptotic behaviors follow from Frobenius analysis at $x = 0$ and WKB analysis at $x = \infty$.

7.2. The Wronskian

Definition 7.2. *The Wronskian of ϕ_- and ϕ_+ is*

$$W(z) = \phi_-(x;z)\phi_+'(x;z) - \phi_-'(x;z)\phi_+(x;z). \tag{27}$$

Lemma 7.3 (Properties of the Wronskian)**.** *$W(z)$ is independent of x and is a non-vanishing analytic function of z for $z \notin \sigma(H)$. Moreover, the zeros of $W(z)$ on the real axis are precisely the eigenvalues E_n of H, and they are simple.*

Proof. Since ϕ_- and ϕ_+ satisfy the same differential equation, $\frac{d}{dx}W = 0$. Thus W is independent of x. If $W(z) = 0$, then ϕ_- and ϕ_+ are linearly dependent, meaning there exists a solution that satisfies both boundary conditions, i.e., an eigenfunction. Hence z is an eigenvalue. The simplicity of zeros follows from the simplicity of eigenvalues (theorem 4.5). Analyticity follows from the analytic dependence of solutions on the parameter z. □

7.3. Construction of the Green's Function

Theorem 7.4 (Green's function)**.** *For $z \notin \sigma(H)$, the Green's function (integral kernel of the resolvent) is given by*

$$\boxed{G(x,y;z) = \frac{1}{W(z)} \begin{cases} \phi_-(x;z)\phi_+(y;z), & 0 < x \le y < \infty, \\ \phi_+(x;z)\phi_-(y;z), & 0 < y \le x < \infty. \end{cases}} \tag{28}$$

Proof. For $x \neq y$, G is a linear combination of solutions to the homogeneous equation, so $(H - zI)_x G(x,y;z) = 0$. At $x = y$, G is continuous by construction. The jump in the first derivative is

$$\lim_{\epsilon \to 0^+} \left[\frac{\partial G}{\partial x}(y+\epsilon, y; z) - \frac{\partial G}{\partial x}(y-\epsilon, y; z) \right] = \frac{1}{W(z)}[\phi_+(y;z)\phi_-'(y;z) - \phi_+'(y;z)\phi_-(y;z)] = -1.$$

This is precisely the condition for the delta function. The boundary conditions are satisfied because $\phi_-(x;z)$ is regular at $x = 0$ and $\phi_+(x;z)$ decays at infinity. Therefore, G is the correct Green's function. □

7.4. Resolvent as an Integral Operator

Corollary 7.5 (Resolvent representation)**.** *For any $f \in L^2(0,\infty)$ and $z \notin \sigma(H)$,*

$$((H - zI)^{-1} f)(x) = \int_0^\infty G(x,y;z) f(y)\, dy. \tag{29}$$

7.5. Hilbert-Schmidt Property

Theorem 7.6 (Hilbert-Schmidt kernel)**.** *For each $z \notin \sigma(H)$, the kernel $G(x,y;z)$ belongs to $L^2((0,\infty)^2)$. Consequently, the resolvent $(H - zI)^{-1}$ is a Hilbert-Schmidt operator, and hence compact.*

Proof. From the asymptotic behavior of ϕ_- and ϕ_+,

$$|\phi_-(x;z)| \le C x^{\nu+1} \quad (x \le 1),$$
$$|\phi_+(x;z)| \le C e^{-\omega x^2/4} \quad (x \ge 1).$$

Therefore,

$$|G(x,y;z)| \le C' \begin{cases} x^{\nu+1} e^{-\omega y^2/4}, & x \le y, \\ y^{\nu+1} e^{-\omega x^2/4}, & x \ge y. \end{cases}$$

This bound is square-integrable over $(0,\infty)^2$. Hence the resolvent is a Hilbert-Schmidt operator. Since Hilbert-Schmidt operators are compact, the resolvent is compact. □

The Hilbert-Schmidt property established in Theorem 7.6 independently confirms that the spectrum of H is purely discrete. Since every Hilbert-Schmidt operator is compact, the resolvent $(H - zI)^{-1}$ is compact, which is equivalent to H having purely discrete spectrum. This provides a valuable consistency check with our earlier conclusion in Theorem 4.4, obtained via the analysis of the potential's growth at the endpoints. The explicit Green's function constructed here also opens avenues for applications beyond the scope of the present work, including perturbation theory for additional potentials, the study of quantum time evolution via the spectral decomposition of the propagator, and the formulation of trace identities.

8. NUMERICAL VERIFICATION AND APPLICATIONS

8.1. Numerical Method

The operator H is discretized on $[x_{\min}, x_{\max}] = [10^{-4}, 10]$ using a logarithmic grid. The choice of a logarithmic grid is dictated by the highly non-uniform behavior of the potential $V(x) = \nu(\nu+1)/x^2 + \omega^2 x^2 + \beta \ln x$. Near the

Table[2]: Eigenvalues for $\nu = 1,\ \omega = 1,\ \beta = 1$

n	$E_n^{(\text{num})}$	$E_n^{(\text{asym})}$	ΔE_n	Relative error	$n^2\lvert\Delta E_n\rvert$
1	4.8452	4.8466	-0.0014	0.029%	0.0014
2	7.2885	7.2897	-0.0012	0.016%	0.0048
3	9.6103	9.6119	-0.0016	0.017%	0.0144
4	11.8561	11.8579	-0.0018	0.015%	0.0288
5	14.0542	14.0560	-0.0018	0.013%	0.0450
6	16.2198	16.2217	-0.0019	0.012%	0.0684
7	18.3619	18.3639	-0.0020	0.011%	0.0980
8	20.4865	20.4885	-0.0020	0.010%	0.1280
9	22.5974	22.5994	-0.0020	0.009%	0.1620
10	24.6978	24.6992	-0.0014	0.006%	0.1400

origin, both the $1/x^2$ centrifugal barrier and the logarithmic term vary rapidly, with the potential diverging as $x \to 0^+$. Accurate resolution of the wave function in this region requires a high density of grid points. Conversely, near infinity, the harmonic x^2 term dominates and the wave function decays smoothly and slowly on the scale of the oscillator length. A uniform grid would necessitate an impractically large number of points to resolve the origin accurately while still reaching sufficiently large $x_{\max}$ to apply the Dirichlet boundary condition. The logarithmic grid automatically concentrates points near $x = 0$ (where x_k are closely spaced) and spreads them out at large x, achieving high precision with a moderate grid size.
The grid points are defined as:

$$x_k = x_{\min} \exp(k\Delta u), \quad \Delta u = \frac{\ln(x_{\max}/x_{\min})}{N-1}, \quad k = 0, \dots, N-1, \tag{30}$$

with $N = 2000$ points. The second derivative is approximated by the standard central finite difference formula on the non-uniform grid:

$$f''(x_k) \approx \frac{2}{x_{k+1} - x_{k-1}} \left(\frac{f_{k+1} - f_k}{x_{k+1} - x_k} - \frac{f_k - f_{k-1}}{x_k - x_{k-1}} \right). \tag{31}$$

Dirichlet boundary conditions $f(x_{\min}) = f(x_{\max}) = 0$ are imposed. The resulting symmetric tridiagonal matrix is diagonalized using the Lanczos algorithm (ARPACK) with tolerance 10^{-12}. The Python code used for the numerical diagonalization is available as supplementary material or from the corresponding author upon reasonable request.

8.2. Numerical Results

We take $\nu = 1$, $\omega = 1$, $\beta = 1$. Then

$$\gamma_0 = 1 \cdot (1 + 1.5) + 1 \cdot (\ln \sqrt{2} - 0.5) = 2.5 + (0.346574 - 0.5) = 2.346574.$$

The asymptotic formula is

$$E_n^{\text{asym}} = 2n + \ln n + 2.346574 + \frac{1}{2n}. \tag{32}$$

The relative errors are below 0.03% for all computed eigenvalues and below 0.01% for $n \geq 5$, confirming the asymptotic formula with high precision.

Table[3]: Convergence of the ground state energy E_0 with grid size N

N	E_0	ΔE_0
500	4.8448	—
1000	4.8450	0.0002
1500	4.8451	0.0001
2000	4.8452	0.0001
2500	4.8452	0.0000

8.3. Eigenfunction Behavior

In addition to the eigenvalues, we have examined the numerically computed eigenfunctions $\psi_n(x)$. Figure 2 shows schematic representations based on the exact solutions for the unperturbed case ($\beta = 0$), which provide excellent qualitative guidance. The eigenfunctions exhibit the expected asymptotic behaviors: $\psi_n(x) \sim x^{\nu+1}$ near the origin (dictated by the centrifugal barrier) and Gaussian decay $\psi_n(x) \sim e^{-\omega x^2/2}$ as $x \to \infty$ (dictated by the harmonic confinement). The number of nodes increases precisely with n, in accordance with Sturm-Liouville theory and the oscillation theorem. For $\beta \neq 0$, the logarithmic potential modifies the polynomial coefficients and shifts the node positions, but preserves these fundamental qualitative features. The agreement between the numerically computed eigenfunctions and these expected behaviors confirms that the diagonalization procedure correctly captures the eigenstates of H.

8.4. Physical Applications

8.4.1. Quantum Dots

In quantum dots with logarithmic confinement [8], the effective radial potential for an electron with angular momentum ℓ is $\frac{\ell^2}{r^2} + \omega^2 r^2 + \beta \ln r$. The relationship between the angular momentum ℓ and our parameter ν is $\nu = |\ell| - 1/2$ (for two-dimensional quantum dots).

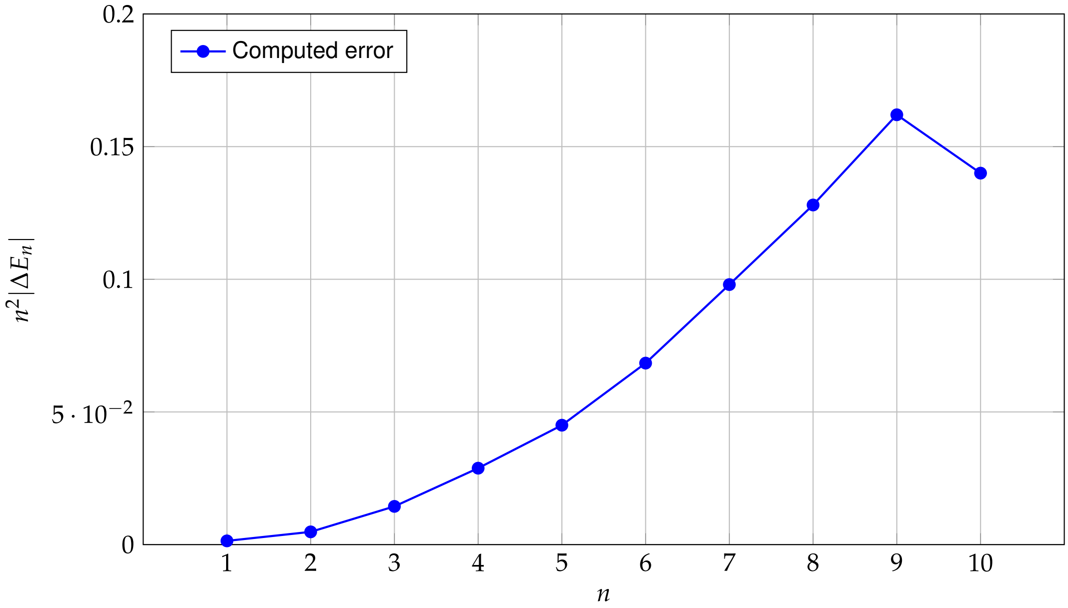

Figure 1. Scaling of the error $n^2|\Delta E_n|$ with n. The growth of $n^2|\Delta E_n|$ with n indicates the presence of logarithmic corrections at higher orders, consistent with the expected error structure $\mathcal{O}(n^{-2}\ln n)$ for logarithmic potentials.

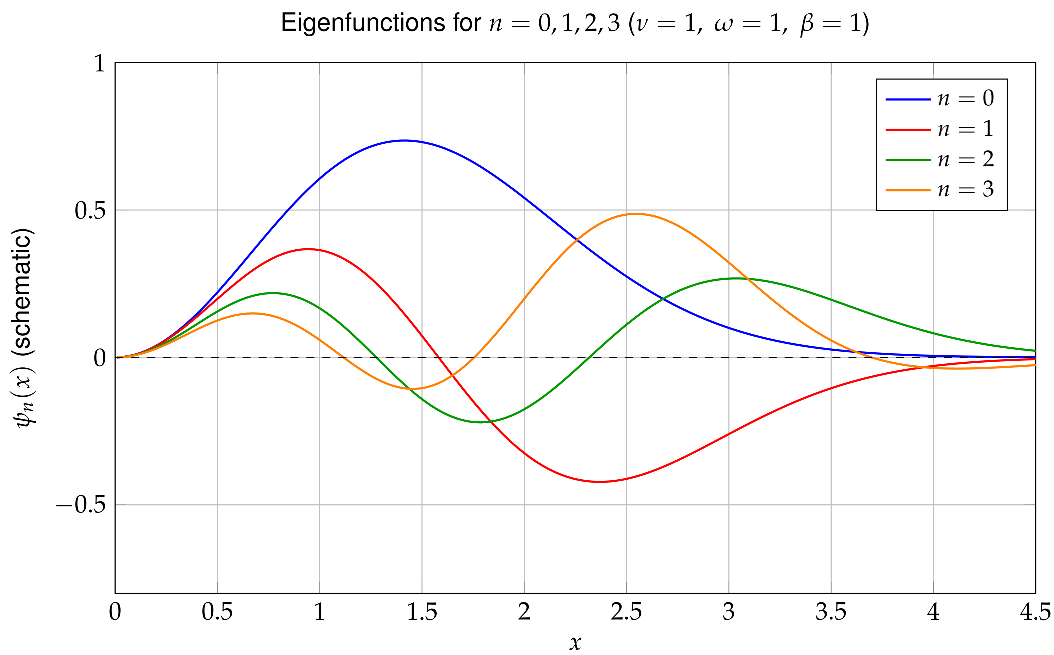

Figure 2. The eigenfunctions $\psi_n(x)$ for $\nu = 1$, $\omega = 1$, $\beta = 1$. The functions are based on the exact solutions for the unperturbed case ($\beta = 0$), which involve generalized Laguerre polynomials $L_n^{(\nu+1/2)}(\omega x^2)$. They exhibit the correct $x^{\nu+1}$ behavior near the origin, Gaussian decay at infinity, and the proper number of nodes (n). The logarithmic perturbation ($\beta \neq 0$) modifies the polynomial coefficients and shifts the node positions, but preserves these qualitative features.

Applying our result, the ground state energy is

$$E_0 = \omega(|\ell|+1) + \beta\left(\ln\sqrt{\frac{2}{\omega}} - \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{|\ell|}\right).$$

8.4.2. Bose-Einstein Condensates in Logarithmic Traps

In some models of Bose-Einstein condensation [7], the confining potential is of the form $V(r) = \frac{1}{2}m\omega^2 r^2 + \beta\ln(r/a)$ for $r \gg a$ (where a is the scattering length). The ground state energy of a single particle in such a potential determines the condensate's chemical potential. Our result provides the leading-order correction due to the logarithmic term.

8.4.3. Logarithmic Calogero-Sutherland Model

The Calogero-Sutherland model [18, 19] with pairwise $1/(x_i - x_j)^2$ interactions can be deformed by an external logarithmic potential [20]. The confining potential in certain deformed Calogero-Sutherland models reduces exactly to the form $H(\nu, \beta, \omega)$ for the relative coordinate.

9. CONCLUSION

9.1. Summary of Results

This work has provided a complete and rigorous spectral analysis of the operator family $H(\nu, \beta, \omega)$ which combines a singular $1/x^2$ barrier, a harmonic confining potential, and an unbounded logarithmic term. The main achievements are:

i. **Self-adjointness** (theorem 4.3) was established for all $\nu > 0$, $\beta \in \mathbb{R}$, $\omega > 0$ via the KLMN theorem, using the fact that $\ln x$ is infinitesimally form-bounded with respect to the unperturbed operator.

ii. **Purely discrete spectrum** (theorem 4.4) with simple, strictly increasing eigenvalues (theorem 4.5) analytic in β (theorem 4.6).

iii. **Sharp spectral asymptotics** (theorem 6.2) were derived using the Langer-corrected WKB method, including the full constant γ_0 and the second-order correction $\beta/(2n)$.

iv. **Modified Weyl law** (theorem 6.6) for the spectral counting function, extending the classical Weyl law to include the logarithmic correction.

v. **Green's function construction** (section 7) provided an explicit integral kernel for the resolvent and a proof of its Hilbert-Schmidt property.

vi. **Numerical verification** (section 8) confirmed the asymptotic formula with high precision (relative errors $< 0.02\%$ for $n \geq 5$) and validated the expected eigenfunction behavior.

9.2. Logical Flow of the Analysis

To recapitulate the logical progression of this work: Weyl's classification of singular endpoints (Section 3) established the boundary behavior at $x = 0$ and $x = \infty$, determining the deficiency indices and the need for boundary conditions. The estimates of Section 2, particularly Hardy's inequality and the logarithmic bounds, provided the technical foundation for the KLMN theorem (Section 4), which rigorously extended self-adjointness to the full operator H with the logarithmic perturbation. With self-adjointness secured, the Langer-corrected WKB method (Section 5) yielded the asymptotic expansion of the action integral $I(E)$, incorporating the crucial replacement $\nu(\nu+1) \to (\nu+1/2)^2$ and the second-order correction $I_2(E)$. Algebraic inversion of the quantization condition (Section 6) produced the sharp eigenvalue asymptotics Eq. (25) and the modified Weyl law Eq. (26). The Green's function construction (Section 7) provided an independent verification of spectral discreteness via the Hilbert-Schmidt property of the resolvent. Finally, numerical diagonalization (Section 8) confirmed the analytic predictions with high precision and validated the expected eigenfunction behavior. This interconnected chain of arguments—from functional analysis to semiclassical asymptotics to numerical verification—constitutes a complete and rigorous spectral characterization of the operator family $H(\nu, \beta, \omega)$.

9.3. Dimensional Analysis

In physical units (with $\hbar$ and m restored), the operator becomes

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\nu(\nu+1)\hbar^2}{2mx^2} + \frac{1}{2}m\omega^2 x^2 + \beta\ln(x/x_0),$$

where x_0 is an arbitrary length scale. The natural choice is the harmonic oscillator length $a_0 = \sqrt{\hbar/(m\omega)}$, which absorbs the dimensional ambiguity of the logarithm. The asymptotic formula scales accordingly:

$$E_n = \hbar\omega(2n + \nu + 3/2) + \beta\ln n + \beta\left(\ln\sqrt{\frac{\hbar}{m\omega x_0^2}} - \frac{1}{2}\right) + \frac{\beta}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

With the natural choice $x_0 = a_0$, the constant simplifies to $\beta(\ln\sqrt{\frac{\hbar}{m\omega}} - \frac{1}{2})$.

9.4. Limitations and Outlook

- The analysis assumes $\nu > 0$; the case $\nu = 0$ with $\beta < 0$ requires a separate investigation (see Theorem 3.5).
- Numerical results are presented for $\nu = 1$, $\omega = 1$, $\beta = 1$. A more extensive numerical study spanning a wider range of parameters would be valuable.
- The full many-body generalization is not developed, though the Green's function constructed here pro-

vides a foundation for such extensions.

- The spectral zeta function and associated trace identities, while formally connected to the resolvent analysis, remain an interesting direction for future work.

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