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Axiomatic of Neutrosophic Groups

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ABSTRACT:
The aim of this article is to; present a neutrosophic group according to axioms such as classical group theory and the neutrosophic set, and to study some properties and theorems related to the neutrosophic group. The new concept of the neutrosophic set is a new approach that is suitable for mathematical problems related to philosophical concepts, such as uncertainty and indeterminacy, in which human knowledge and human evaluation are necessary. Neutrosophic algebra is a branch of neutrosophic set theory, and in 2004, Kandasamy and Smarandache introduced basic neutrosophic algebraic structures and their applications to fuzzy and neutrosophic models; in 2006, Kandasamy and Smarandache presented neutrosophic algebraic structures and neutrosophic N-algebraic structures; in 2019, Smarandache introduced new fields of research in Neutrosophy, which he called Neutro-Structures and Anti-Structures. In 2020, Agboola presented an Introduction to the neutrosophic group by a different presentation.

CONTENTS:

1. INTRODUCTION

The neutrosophic group is a branch of neutrosophic algebra, which is a branch of Neutrosophy, Neutrosophy, a branch of new philosophy proposed by Smarandache with the development and extension of the intuitionistic fuzzy set into the neutrosophic set that is associative with the neutrosophic logic, for more information about the neutrosophic set and the neutrosophic logic, we refer to [1], [2], [3], and [4]. The concept of indeterminacy " I ", where $I^2 = I$ vis. The concept of imagery in complex

numbers $j^2 = -1$, and consequently, when " I " cannot be defined. In 2006 [5], and [6], Kandasamy published Smarandache neutrosophic algebraic structures including neutrosophic groups and their properties, as well as other structures of neutrosophic algebras; and presenting neutrosophic groups and studied Neutrosophic Bi-groups with their properties, Neutrosophic N-groups with their properties, and other structures of Neutrosophic algebra with Smarandache. Previous studies urged (or motivated) researchers to study classical algebraic structures from the perspective of neutrosophic theories. In 2012 [7], Agboola, Akwu, and Oyebo studied neutrosophic groups and subgroups, and followed their work in 2019, and 2020 [8], [9] with other researchers by presenting the product of a neutrosophic subgroup and a pseudo neutrosophic subgroup of a commutative neutrosophic group is a neutrosophic subgroup and their union is also a neutrosophic subgroup. They follow the condintional argument due to Smarandache, If G is a group, then $\langle G \cup I \rangle$ is a netrusphic group, where, $G\{I\} = \{a + bI : a; b \in R \,\&\, I^2 = I, 0I = 0\}$. In 2023, there were some results about neutrosophic groups according to the degree of neutrosophic membership function [10]. In this article, we introduce our contribution to Neeutrosphic theory, in particular, by introducing the axiomatic Neutrosophic group by a different path, namely, we consider the neutrosophic set is given by:
$G\{I\} = \{a + bI : a; b \in R \,\&\, I^2 = I, 0I = 0\}$ and define the binary operations on $G\{I\}$. In other words, when the degree of membership is fully true, and the binary operation satisfies the axiomatic group, study some theorems and properties with examples. This work, Which is a treatment with just a neutrosophic set only, is relevant to our work in [11], [12], and [13].

2. EUTROSPHIC GROUPS AND THEIR PROPERTIES

In this section, we introduce the concept of neutrosophic groups introduced in 2006 by Kandasamy and Smarandache [5] and [6], we presenting the neutrosophic groups according to axiomatic classical group theory.

Definition 2.1 [5,6] Let $(G,*)$ be any group, and $\langle G \cup I \rangle$ is given by:
$\langle G \cup I \rangle = \{a + bI : a, b \in G\}$,
then the neutrosophic algebra structure $N(G) = \{\langle G \cup I \rangle, *\}$ is called the neutrosophic group which is a generated by I and G under $*$.

Theorem 2.1 [5,6] Let $(G,*)$ be a group, $N(G) = \{\langle G \cup I \rangle, *\}$ be the neutrosophic group, then:

1. $N(G)$ in general, is not a group, and
2. $N(G)$ always contains a group.

Definition 2.2 Let $G \neq \emptyset$ be any non-empty set and $G[I] = \{a + bI : a, b \in G\}$ be called the neutrosophic set (NS). The order pair $NG = \langle G[I], * \rangle$ consists of the neutrosophic set $G[I]$ with binary operation $*$ on $G[I]$ and is called the Neutrosophic Group (NG); if it satisfies the following axioms:

NG$_1$: For all x ,y and $z \in NG$, $(x * y) * z = x * (y * z)$ " associative law";
NG$_2$: There exists $e_N = e + eI \in NG$ such that for all $x * e_N = x = e_N * x$ " existence of an identity" and;
NG$_3$: For all $x \in G[I]$, there exists $y \in G[I]$ such that $x * y = e_N = y * x$ " existence of inverse. " Thus, the neutrosophic group is a neutrosophic mathematical system.
$NG = \langle G[I], * \rangle$ satisfying the axioms NG$_1$ to NG$_3$. Otherwise is called a neutrosophic algebra structure.

Definition 2.3 Let $N(G) = \langle G[I],*\rangle$ be a neutrosophic group; then, it is said to be a commutative neutrosophic group if for all $x, y \in G[I]$, $x * y = y * x$, otherwise it is called a non-commutative neutrosophic group.

Definition 2.4 If G is a finite set,then the neutrosophic set $G[I]$ is finite and the neutrosophic group $N(G) = \langle G[I],*\rangle$ is called finite; otherwise, it is called the infinite neutrosophic group. The number of netrosphic elements in $G[I]$ is called the order of the neutrosophic group, and is denoted by $O(G[I])$.
The folloing theeorems give us the main properties of neutrosophic group:
$N(G) = \langle G[I],*\rangle$.

Theorem 2.2 Let $N(G) = \langle G[I],*\rangle$ be a neutrosophic group.

1. There exists a unique element $e_N = e + eI \in G[I]$ such that $x * e = x = e * x$, for all $x \in G[I]$.
2. There exists a unique $y \in N(G)$ such that $x * y = y * x = e$.

Proof.

1. By NG2, there exists $e = e + eI \in G[I]$such that for all $x * e = x = e * x$, for all $x \in G[I]$.suppose that $e' = e' + e'I \in G[I]$ be another identity element, to show that $e = e'$. Since e is identity, we have $e * e' = e'$ and $e' * e = e'$ (1). In addition, because e' is an identity, we have $e * e' = e$ and $e' * e = e$ (2). From (1) and (2), we obtaning,
$$e' = e' * e = (e' + e'I) * (e + eI) = \Big((e' * e) + \big((e' * e)I\big)\Big) = (e + eI) = e.$$
Hence the neutrosophic identity of element is a unique.
2. Let $x \in G[I]$, and by NG3, there exists $y \in G[I]$ such that $x * y = y * x = e$. Suppose that there exists $z \in G[I]$ exists such that $x * z = e = z * x$. We need to show that $y = z$,now,
$y = y * e = y * (x * z)$
$= (y * x) * z$
$= e * z = z$; thus, y is a unique.

Theorem 2.3. Let $N(G) = \langle G[I],*\rangle$ be a neutrosophic group. Then:

1. $((a + bI)^{-1})^{-1} = (a + bI)$, for all $(a + bI) \in G[I]$.
2. $\big((a + bI) * (c + dI)\big)^{-1} = (c + dI)^{-1} * (a + bI)^{-1}$, for all $(a + bI), (c + dI) \in G[I]$.
3. For all $x, y, z \in N(G)$, if either $x * z = y * z$ or $z * x = z * y$, then $x = y$, this is called the Cancelation law in the neutrosophic group.
4. For all $x, y \in G[I]$, the unique solution $x * z = y$ has a unique solution in $N(G)$ for z.

Proof.

1. Let $x = (a + bI) \in G[I]$, By GN3, there exists inverse
$x^{-1} = (a^{-1} + b^{-1}I) \in G[I]$ such that
$x * x^{-1} = (a + bI) * (a^{-1} + b^{-1}I)$
$= \big((a * a^{-1}) + (b * b^{-1})I\big)$
$= (e + eI) = e_N$, Simultaneously, we have;

$x^{-1} * x = (a^{-1} + b^{-1}I) * (a + bI)$
$= \left((a^{-1} * a) + (b^{-1} * b)I\right)$
$= (e + eI) = e_N$, since e_N is unique by theorem 2.2. Hence, $(x^{-1})^{-1} = ((a^{-1})^{-1} + (b^{-1})^{-1}I) \in G[I]$ denote the inverse of $x^{-1} = (a^{-1} + b^{-1}I)$, we deduced that
$(x^{-1})^{-1} = ((a^{-1})^{-1} + (b^{-1})^{-1}I) = x = (a + bI)$.

2. $(x * y) * (y^{-1} * x^{-1}) = \left((x * y) * y^{-1}\right) * x^{-1}$
$= \left((a + bI) * (c + dI) * (c^{-1} + d^{-1}I)\right) * x^{-1}$
$= \left([(a * c) + (b * d)I] * (c^{-1} + d^{-1}I)\right) * x^{-1}$
$= \left((a * c) * c^{-1} + ((b * d) * d^{-1})I\right) * x^{-1}$
$= (a * (c * c^{-1}) + (b * (d * d^{-1})I) * x^{-1}$
$= (a * e + (b * e)I) * x^{-1}$
$= (a + bI) * (a^{-1} + b^{-1}I)$
$= (a * a^{-1} + (b * b^{-1})I)$
$= (e + eI) = e_N$. By similar argument, we have,
$(y^{-1} * x^{-1}) *). (x * y) = e_N$. Therefore $(y^{-1} * x^{-1})$ is a neutrosophic inverse of $(x * y)$, since the inverse of neutrosophic element in neutrosophic group is a unique, it follows that
$\left((a + bI) * (c + dI)\right)^{-1} = (c + dI)^{-1} * (a + bI)^{-1}$.

3. Let $x = (a + bI), y = (c + dI)$ and $z = m + nI \in G[I]$. Suppose that:
$x * z = y * z \Rightarrow (x * z) * z^{-1} = (y * z) * z^{-1}$
$\Rightarrow x * (z * z^{-1}) = y * (z * z^{-1})$
$\Rightarrow x * e_N = y * e_N$
$\Rightarrow x = y$. By similar manner, if $z * x = z * y$, then $x = y$.

4. Let $x = (a + bI), y = (c + dI) \in G[I]$, initialy, we consider the equation $x * z = y$, because $x \in N(G)$, by NG$_2$, we have $x^{-1} \in G[I]$and consequently, $x^{-1} * y \in G[I]$.Now, take L.H.S:
$x * z = x * (x^{-1} * y) = (x * x^{-1}) * y = e_N * y = y$. Hence $x^{-1} * y$ is a solution of the equation $x * z = y$. To prove that uniqueness of the solution. Assume that m is another solution of $x * z = y$. That is, $x * m = y$.
To show that $z = m$.
$m = e_N * m = (e + eI) * (m_1 + m_2I)$
$= \left((a + bI)^{-1} * (a + bI)\right) * (m_1 + m_2I)$
$= (a + bI)^{-1} * \left((a + bI) * (m_1 + m_2I)\right)$.
$= (a + bI)^{-1} * \left((c + dI)\right) = z$. Hence the solution is unique.

3. SOME TYPES OF NEUTROSPHIC GROUPS

3.1. Some Types of Infinte Neutrosophic Groups

Definition 3.1.1 [5,6] Let $\mathbb{Z}$ be a set of integer numbers, and $\mathbb{Z}[\mathrm{I}] = \{a + bI : a, b \in \mathbb{Z}\}$ be a neutrosophic integer set, where $a + bI$ is a neutrosophic integer number.

Proposition 3.1.1 Let $\mathbb{Z}[I] = \{a + bI: a, b \in \mathbb{Z}\}$ be the set of netrosophic integer numbers. Then the netrosophic strcuture $N(\mathbb{Z}) = \langle \mathbb{Z}[I], + \rangle$ under usual addition forms a commuatative netrosophic integer group.

Proof. Let $a = a_1 + a_2 I$, $b = b_1 + b_2 I$ and $c = c_1 + c_2 I$ be the three elements of $\mathbb{Z}[\mathrm{I}]$. It is clear that $+$ is a binary opearation of $\mathbb{Z}[I]$.

NG1: $(a + b) + c = \big((a_1 + a_2 I) + (b_1 + b_2 I)\big) + (c_1 + c_2 I)$
$= ((a_1 + b_1) + (a_2 + b_2)I) + (c_1 + c_2 I)$
$= [((a_1 + b_1) + c_1) + (a_2 + b_2) + c_2)I]$
$= [(a_1 + (b_1 + c_1)) + (a_2 + (b_2 + c_2)I]$
$= [(a_1 + (b_1 + c_1) + (a_2 + b_2) + c_2)I]$
$= (a_1 + a_2 I) + \big((b_1 + b_2 I) + (c_1 + cI)\big)$
$= a + (b + c)$. Hence it is associative.

NG2: $\exists\, 0 = 0 + 0.I \in N(\mathbb{Z})$ such that
$0 + a = (0 + 0.I) + (a_1 + a_2 I)$
$= \big((0 + a_1) + (0 + a_2)I)\big)$
$= (a_1 + a_2 I) = a$. By similar method $a + 0 = a$. So $e = 0 = 0 + 0.I$

NG3: $\forall\, a = a_1 + a_2 I \in N(\mathbb{Z}) \Rightarrow \exists\, a^{-1} = (-a) = (-a_1) + (-a_2)I$ such that:
$a + (-a) = (a_1 + a_2 I) + (-a_1) + (-a_2)I$
$= (a_1 + -a_1) + (a_2 + -a_2)I) = 0 + 0.I = 0$,

Also $(-a) + a = 0$. We deduced that $N(\mathbb{Z}) = \{\langle \mathbb{Z} \cup I \rangle, +\}$ is a neutrosophic group, in addition,
$a + b = (a_1 + a_2 I) + (b_1 + b_2 I)$
$= ((a_1 + b_1) + (a_2 + b_2)I)$
$= ((b_1 + a_1) + (b_2 + a_2)I)$
$= (b_1 + b_2 I) + (a_1 + a_2 I)$
$= b + a$, that is it is the commutative neutrosophic group.

Proposition 3.1.2 Let $\mathbb{Z}[I] = \{a + bI: a, b \in \mathbb{Z}\}$ be the set of netrosophic integer numbers; then, the netrosophic structure $N(\mathbb{Z}) = \langle \mathbb{Z}[I], \bullet \rangle$ under usual multiplication forms a commuatative netrosophic integer moniod.

Proof. Let $a = a_1 + a_2 I$, $b = b_1 + b_2 I$ and $c = c_1 + c_2 I$ be the three elements of $\mathbb{Z}[\mathrm{I}]$. It is evident that $\bullet$ is a binary opearation on $\mathbb{Z}[I]$. Since,
$a \bullet \mathrm{b} = (a_1 + a_2 I) \bullet (b_1 + b_2 I)$
$= \big(a_1 \bullet b_1 + ((a_1 \bullet b_2) + (a_2 \bullet b_1) + (a_2 \bullet b_2))I\big) \in \mathbb{Z}[\mathrm{I}]$.

NG1: $(a \bullet b) \bullet c = \big((a_1 + a_2 I) \bullet (b_1 + b_2 I)\big) \bullet (c_1 + c_2 I)$
$= \big(a_1 \bullet b_1 + ((a_1 \bullet b_2) + (a_2 \bullet b_1) + (a_2 \bullet b_2))I\big) \bullet (c_1 + c_2 I)$
$= [((a_1 + b_1) + c_1) + (a_2 + b_2) + c_2)I]$
$= [(a_1 + (b_1 + c_1)) + (a_2 + (b_2 + c_2)I]$
$= [(a_1 + (b_1 + c_1) + (a_2 + b_2) + c_2)I]$
$= (a_1 + a_2 I) + \big((b_1 + b_2 I) + (c_1 + cI)\big)$
$= a + (b + c)$. Hence it is associative.

Definition 3.1.2 [5,6,13**]** Let $\mathbb{R}$ be a set of real numbers, and
$\mathbb{R}[I] = \{a + bI: a, b \in \mathbb{R}\}$ be a neutrosophic- real set, where $a + bI$ is a neutrosophic of a real number.

Proposition 3.1.3 Let $\mathbb{R}[I] = \{a + bI: a, b \in \mathbb{R}\}$ be the set of netrosophic real numbers. Then the structure $N(\mathbb{R}) = \langle \mathbb{R}[I], + \rangle$ under the usual addition form a commuatative netrosophic real group.
Proof. By the same techniques of preposition 3.1.1.

Definition3.1.3 [5,6,13**]** Let $\mathbb{C}[I] = \{a + bI: a, b \in \mathbb{C}\}$ be the set of neutrosophic complex numbers, where $a + bI$ is the neutrosophic of a complex number.
Proposition 3.1.4 Let $\mathbb{C}[I] = \{a + bI: a, b \in \mathbb{R}\}$ be the set of neutrosophic complex numbers. Then the neutrosophic algebra structure $N(\mathbb{C}) = \langle \mathbb{C}[I], + \rangle$ under the usual addition forms the commuatative neutrosophic complex group.
Proof. Let $a = a_1 + a_2 I$, $b = b_1 + b_2 I$ and $c = c_1 + c_2 I$ be three elements in $\mathbb{C}[I]$.
We have:

$$\begin{aligned} a + b &= (a_1 + a_2 I) + (b_1 + b_2 I) \\ &= [(a_1 + b_1) + (a_2 + b_2)I] \\ &= (a' + a'_1 i + b' + b'_1 i) + \big((a'' + a'_2 i) + (b' + b'_2 i)I\big) \\ &= \Big((a' + b') + (a'_1 + b'_1)\ i\Big) + \big((a'' + b')I) + (a'_2 i + b'_2 i)I\big). \end{aligned}$$ It is a closure.

NG$_1$: $$\begin{aligned} (a+b)+c &= \left(\begin{array}{c} \Big((a' + b') + (a'_1 + b'_1)\ i\Big) + \big((a'' + b')I) + (a'_2 i + b'_2 i)I\big) \\ + \\ (c' + c'_1 i) + (c'' I + c'_2 iI) \end{array}\right) \\ &= \left(\begin{array}{c} \big((a' + b') + c' + (a'_1 + b'_1) + c'_1)\ i\big) \\ + \\ \big((a'' + b') + c'')I) + ((a'_2 i + b'_2 i) + c'_2 i)I\big) \end{array}\right) \\ &= \left(\begin{array}{c} (a' + (b' + c') + (a'_1 + (b'_1 + c'_1)\ i) \\ + \\ (a'' + (b' + c'')I + (a'_2 i + (b'_2 i + c'_2 i)I) \end{array}\right) \end{aligned}$$, it is clear that:
$= a + (b + c)$.

NG$_2$: $\exists\ 0 = (0 + 0.i) + (0.I + 0.i.I) \in N(\mathbb{C})$ such that:
$$\begin{aligned} 0 + a &= (0 + 0.I) + (a_1 + a_2 I) \\ &= \big((0 + a') + (0 + a'_1)i + (0. + a')I) + (0.i + a'_2.i)I\big) = a. \end{aligned}$$ By similar method $a + 0 = a$.

NG$_3$: $\forall\ a = a_1 + a_2 I = (a' + a'_1 i) + (a'' I + a'_2 iI) \in N(\mathbb{C})$, then there exists,
$a^{-1} = (-a) = (-a_1) + (-a_2)I = (-a' + -a'_1 i) + (-a'' I + -a'_2 iI) \in N(\mathbb{C})$, evidently, satisfy the condition: $a + (-a) = 0$ and $(-a) + a = 0$.

3.2. Finite Neutrosophic Integer Groups of Modulation

Definition 3.2.1 [15] Let $\mathbb{Z}(I) = \{a + bI: a, b \in \mathbb{Z}\}$ be a neutrosophic set of integers. We define a congruence relation $\equiv$ on $\mathbb{Z}(I)$ as follows:
$\forall\ x, y \in \mathbb{Z}(I), x \equiv y (mod\ z) \Leftrightarrow \exists\ z \in \mathbb{Z}(I)$ such that $z | x - y$.

Theorem 3.2.1 [15] Let $\mathbb{Z}(I) = \{a + bI: a, b \in \mathbb{Z}\}$ be the neutrosophic set of integers. Consider: $x = a + bI, y = c + dI$, and $z = m + nI$, then $x \equiv y(mod\ z) \Leftrightarrow a \equiv c(mod\ m)$ and $a + b \equiv c + d(mod\ m + n)$.

Theorem 3.2.2 The congruence relation $\equiv$ on $\mathbb{Z}(I)$ is an equivalence relation.
Proof.
Consider $x = a + bI, y = c + dI$, and $z = m + nI$ in $\mathbb{Z}(I)$.

1. $\equiv$ is reflexive, since $a - a = 0 = 0\ m$ and
   $(a + b) - (a + b) = 0\ (m + n)$, hence $x \equiv x(\ mod\ z)$.
2. $\equiv$ is symmetric, consider $x \equiv y(\ mod\ z) \Rightarrow a \equiv c(mod\ m)$ and
   $a + b \equiv c + d(mod\ m + n)$. Hence $c \equiv a(mod\ m)$ and
   $c + d \equiv a + b(mod\ m + n)$, therefore, $y \equiv x(\ mod\ z)$.
3. $\equiv$ is transitive, Suppose that $x \equiv y(\ mod\ z)$, then
   $a \equiv c(mod\ m)$ and $a + b \equiv c + d(mod\ m + n)$.And
   $y \equiv r(\ mod\ z) \Rightarrow c \equiv r_1(mod\ m)$ and $c + d \ \equiv\ r_1 + r_2(mod\ m + n)$.
   $\Rightarrow\ a \equiv r_1(mod\ m)$ and $a + b \equiv r_1 + r_2(mod\ m + n)$
   $\Rightarrow\ x \equiv r(mod\ m + n)$, so $\equiv$ is an equivalence relation on $\mathbb{Z}(I)$.

Theorem 3.2.2 Let $Z_n[I] = \{[a] + [b]I: [a], [b]\ \in Z_n\}$ be a finite set of intger modulos n. Define $\oplus_n$ on $Z_n[I]$ by:
$[a_1 + b_1I] \oplus_n [a_2 + b_2I] = [(a_1 + a_2) + (b_1 + b_2)I] = [a_3 + b_3I]$ (modulo n). For all $[a_1 + b_1I]$ and $[a_1 + b_1I] \in Z_n[I]$. Then the neutrosophic algebra structure $N(Z_n[I]) = \langle Z_n[I],\oplus_n\rangle$ forms commutative neutrosophic group whis is called the neutrosophic integer modulo n.

Proof. First to show the $\oplus_n$ is a binary operation.

1. Let $x = [a_1 + b_1I], y = [a_2 + b_2I], z = [a_3 + b_3I]$ and $w = [a_4 + b_4I]$ be the four elements of $Z_n[I]$. Suppose that $x = y$ and $z = w$,
   Since $x = y \Rightarrow a_1 \equiv a_2(mod\ m)$ and $a_1 + b_1 \equiv a_2 + b_2(mod\ m + n)$. Also,
   Since $z = w \Rightarrow a_3 \equiv a_4(mod\ m)$ and $a_3 + b_3 \equiv a_4 + b_4(mod\ m + n)$,
   therefore, $a_1 + a_3 \equiv a_2 +\ a_4(mod\ m)$, and
   $(a_1 + b_1) + (a_3 + b_3\ ) \equiv (a_2 + b_2) + (a_4 + b_4)(mod\ m + n)$, hence,
   $x + z \equiv y + w\ (\ modulo\ z)$, so
   $[(a_1 + a_3) + (b_1 + b_3)I] = [(a_2 + a_4) + (b_2 + b_4)I]$ from the conclusion
   $\oplus_n$ is well defined on $Z_n[I]$. For all $[a_1 + b_1I]$ and $[a_1 + b_1I] \in Z_n[I]$,we
   Have,
   $x = [a_1 + b_1I] \oplus_n y = [a_2 + b_2I]$
   $= [(a_1 + a_2) + (b_1I + b_2I)] = [a_3 + b_3I]$ (modulo n).So it is a closure.

2. **NG1:** For all $x = [a_1 + b_1I]$,$y = [a_2 + b_2I]$ and $z = [a_3 + b_3I] \in Z_n[I]$, we have,
   $(x + y) + z = ([a_1 + b_1I] + [a_2 + b_2I]) + [a_3 + b_3I]$
   $= [(a_1 + a_2) + (b_1\ + b_2)I] + [a_3 + b_3I]$
   $= [(a_1 + a_2) + a_3 + ((b_1\ + b_2) + b_3)I]$
   $= [a_1 + (a_2 + a_3)\ + (\ b_1 +\ (b_2\ + b_3))I]$
   $= [a_1 + b_1I] + [(a_2 + a_3)\ +\ (b_2\ + b_3)I]$
   $= x + (y + z)$.Hence associative law is hold.

3. **NG2:**There exists an identity element $0 = [0 + 0I] \in Z_n[I]$,
   such that, $0 + x = [0 + 0I]$+$[a_1 + b_1I]$

$= [(0 + a_1) + (0 + b_1)I]$
$= [a_1 + b_1 I] = x$, for all $x \in Z_n[I]$,

4. **NG3:**For all $x \in Z_n[I]$, , there is an inverse element $x^{-1} \in Z_n[I]$ such that,$x + x^{-1} = [a_1 + b_1 I] + [-a_1 + (-b_1)I]$
$= [(a_1 + (-a_1) + (b_1 + (-b_1)I]$
$= [0 + \ 0.I] = 0$. It is clear that $\oplus_n$ is a commutative on $Z_n[I]$, hence $\langle Z_n[I], \oplus_n \rangle$ is commutative neutrosophic group.

Example 3.2.1 Let $Z_3[I] = \{a + bI : a, b \ \in Z_3\}$
$= \{0,1,2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$ be a netrosphic integers of modulo 3 with binary opearation $\oplus_3$ on $Z_3[I]$. To construct the neutrosophic group $\langle Z_3[I], \oplus_3 \rangle$ modulo 3 using the visualization table as shown in table.3.2.1

Table.3.2.1 of $\langle Z_3[I], \oplus_3 \rangle$.

$\oplus_3$	0	1	2	I	$2I$	$1+I$	$1+2I$	$2+I$	$2+2I$
0	0	1	2	I	$2I$	$1+I$	$1+2I$	$2+I$	$2+2I$
1	1	2	0	$1+I$	$1+2I$	$2+I$	$2+2I$	I	$2I$
2	2	0	1	$2+I$	$2+2I$	I	$2I$	$1+I$	$1+2I$
I	I	$1+I$	$2+I$	$2I$	0	$1+2I$	1	$2+2I$	2
$2I$	$2I$	$1+2I$	$2+2I$	0	I	1	$2+I$	2	$2+I$
$1+I$	$1+I$	$2+I$	I	$1+2I$	1	$2+2I$	2	$2I$	0
$1+2I$	$1+2I$	$2+2I$	$2I$	1	$1+I$	2	$2+I$	0	I
$2+I$	$2+I$	I	$1+I$	$2+2I$	2	$2I$	0	$1+2I$	1
$2+2I$	$2+2I$	$2I$	$1+2I$	2	$2+I$	0	I	1	$1+I$

From the table 3.2.1 it is a closure under operation $\oplus_3$ modulo 3 and associative, there exists identity element is zero and for any elements in x has inverse as shown in the table 3.2.2.

Table 3.2.2. of inverse elements.

x	0	1	2	I	$2I$	$1+I$	$1+2I$	$2+I$	$2+2I$
x^{-1}	0	2	1	$2I$	I	$2+2I$	$2+I$	$1+2I$	$1+I$

$\langle Z_3[I], \oplus_3 \rangle$ is represents a neutrosophic commutative group modulo 3.

Observations. The order of the finite group $(Z_3, \oplus_3)$ is 3 and, and the order of neutrosophic group $\langle Z_3[I], \oplus_3 \rangle$ is 3^2. Hence, the order of a group is divided by that of the neutrosophic group.

Proposition 3.2.3 Let $\mathbb{Z}_p^*[I] = \mathbb{Z}_p[I] \backslash \{0\} = \{[a] + [b]I : [a], [b] \ \in \mathbb{Z}_p^*\}$. be a finite set of intgers modulo p, where p is an aprime number. Define a binary operation $\otimes_p$ On $\mathbb{Z}_p^*[I]$ by:
$x \otimes_p y = [x_1 + x_2 I] \otimes_p [y_1 + y_2 I]$
$= [(x_1.y_1) + ((x_1.y_2) + (x_2.y_1) + (x_2.y_2))I]$(modulo p). For all $x = [x_1 + x_2 I]$ and$y = [y_1 + y_2 I] \in \mathbb{Z}_p^*[I]$. Then $\otimes_p$ is well-defined and assocative with unit elment.

Proof.
Let $x = [x_1 + x_2 I]$, $y = [y_1 + y_2 I]$, $z = [z_1 + z_2 I]$ and $w = [w_1 + w_2 I]$ be four elements belong to $\mathbb{Z}_p^*[I]$. Suppose that $x = y$ and $z = w$,
Since $x = y \Rightarrow x_1 \equiv y_1 (mod\ m)$ and $(x_1 + x_2) \equiv (y_1 + y_2)(mod\ m + n)$. Also,
Since $z = w \Rightarrow z_1 \equiv w_1 (mod\ m)$ and $(z_1 + z_2) \equiv (w_1 + w_2)(mod\ m + n)$, therefore, $(x_1.z_1) \equiv (y_1.w_1)(mod\ m)$ and
$(x_1 + x_2).(z_1 + z_2) \equiv (y_1 + y_2).(w_1 + w_2)(mod\ m + n)$, hence
$x.z \equiv y.w\ (\ modulo\ z)$, so

$$[(x_1 + x_2 I) \otimes_p (z_1 + z_2 I)] = [(y_1 + y_2 I) \otimes_p (w_1 + w_2) I]$$
$$= [(x_1 z_1) + ((x_1.z_2) + (x_2.z_1) + (x_2.z_2) I)]$$
$$= [(y_1 w_1) + ((y_1.w_2) + (y_2.w_1) + (y_2.w_2) I)],$$

therefore from the conclusion $\otimes_p$ is well defined on $\mathbb{Z}_p[I]$. To show that $\otimes_p$ is assocative, for all $x = [x_1 + x_2 I]$,$y = [y_1 + y_2 I]$ and $z = [z_1 + z_2 I] \in Z_n[I]$. We have

$$(x.y).z = ([x_1 + x_2 I] \otimes_p [y_1 + y_2 I]) \otimes_p [z_1 + z_2 I]$$
$$= [(x_1 y_1) + ((x_1.y_2) + (x_2.y_1) + (x_2.y_2) I)] \otimes_p [z_1 + z_2 I]$$
$$= \left[(x_1 y_1).z_1 + \begin{pmatrix} (x_1 y_1).z_2 + ((x_1.y_2) + (x_2.y_1) + (x_2.y_2).).z_1 \\ +((x_1.y_2) + (x_2.y_1) + (x_2.y_2)).z_2 \end{pmatrix} I \right]$$
$$= \left[x_1 (y_1.z_1) + \begin{pmatrix} x_1 (y_1.z_2) + ((x_1.y_2).z_1 + (x_2.y_1).z_1 + (x_2.y_2).z_1) \\ +((x_1.y_2).z_2 + (x_2.y_1).z_2 + (x_2.y_2).z_2) \end{pmatrix} I \right]$$
$$= \left[x_1 (y_1.z_1) + \begin{pmatrix} x_1 (y_1.z_2) + (x_1.(y_2.z_1) + x_2.(y_1.z_1) + x_2.(y_2.z_1)) \\ +(x_1.(y_2.z_2) + x_2.(y_1.z_2) + x_2.(y_2.z_2)) \end{pmatrix} I \right]$$
$$= [x_1 + x_2 I] \otimes_p \left[(y_1.z_1) + \begin{pmatrix} (y_1.z_2) + ((y_2.z_1) + (y_1.z_1) + (y_2.z_1)) \\ +((y_2.z_2) + (y_1.z_2) + (y_2.z_2)) \end{pmatrix} I \right]$$
$$= [x_1 + x_2 I] \otimes_p [(y_1 z_1) + ((y_1.z_2) + (y_2.z_1) + (y_2.z_2) I)]$$
$$= [x_1 + x_2 I] \otimes_p [(y_1 z_1) + ((y_1.z_2) + (y_2.z_1) + (y_2.z_2) I)]$$
$$= [x_1 + x_2 I] \otimes_p \ ([y_1 + y_2 I] \otimes_p [z_1 + z_2 I]) = x.(y.z).$$

Finally, there exists a neutrosophic identity element. $1 = [1 + 0I] \in \langle \mathbb{Z}_n \backslash \{0\}[I], \otimes_p \rangle$ such that,

$$1.x = [1 + 0I] \otimes_n [x_1 + x_2 I]$$
$$= [(1.x_1) + (1.x_2 + 0.x_1 + 0.x_2) I]$$
$$= [x_1 + x_2 I] = x, \forall\ x \in \langle \mathbb{Z}_p \backslash \{0\}[I], \otimes_p \rangle.$$

It is clear that $\{\langle \mathbb{Z}_p[I] \rangle, \otimes_p\}$ is a neutrosophic monoid. Moreover, it is commutative neutrosophic monoid

Example 3.2.2 Let $\mathbb{Z}_3^*[I] = \mathbb{Z}_3[I] \backslash \{0\} = \{[a] + [b]I : [a], [b] \in \mathbb{Z}_3\}$ be be a finite netrsosphic set of intgers modulo 3. Here $\mathbb{Z}_3^* = \{1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$. Under maultiplication the there is no inverse for some elemnts such as I and $\langle \mathbb{Z}_3^*[I], \otimes_3 \rangle$ is a commutative neutrosophic semigroup, but not a group.

4. CONCLUSION

In this article, we introduced the axiomatic of the neutrosophic groups depending on the neutrosophic sets by different arguments of other scholars. Namely, when the degree of memberships of binary operation is 100%, and deduced that some properties and facts that relevant to classical group theory, with concerning the neutrosophic groups.

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