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The Computations of Algebraic Structure of Neutrosophic Determinants

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ABSTRACT
This paper aims to make a valuable contribution to the field of neutrosophic determinants and their properties. By utilizing neutrosophic real numbers in the form of a+bI, we provide an alternative approach to recent research on determinants conducted between 2020 and 2023. Our goal is to expand the scope of academic content being developed in the theory of neutrosophic linear algebra. Additionally, we seek to complement our work on some algebraic structures of neutrosophic matrices.

CONTENTS

1. Introduction:

There are numerous scientific issues in our lives that pose challenges in determining their truth, correctness, falsity, or incorrectness. This difficulty arises due to factors such as limited knowledge, indeterminate concepts, vagueness, and ambiguity inherent in language and the scientific logic used to interpret these issues. Philosophers and mathematicians play a crucial role in interpreting and addressing these indeterminacy issues. In this regard, early works in the field of philosophy have highlighted the necessity for a three-valued logic when dealing with arguments in language that involve future contingent sentences.

In 1920, it was the beginning of triple-value logic with Lukasiewicz. Moreover, Lukasiewicz introduced his future contingent proposition, which read, "I shall be in Warsaw at noon 21 December of the next year." He justified the third logical value of 1/2 as a possibility or indeterminacy. Two years later, namely in 1922, Lukasiewicz generalized many-valued logics, both finite and infinite-valued, and defined the set of logical values of n-valued logic for any natural $n \geq 2$ is

$$L_n = \{0, {}^{1}/_{n-1}, {}^{2}/_{n-1}, {}^{3}/_{n-1}, \dots, {}^{n-2}/_{n-1}, 1\}$$

[4, 13]. After a late period, specifically in 1965, Zadeh in his article proposed the interval [0,1] for the function of a degree membership instead of the set of classical logic {0,1}, despite not referring to Lukasiewicz's work, but he referred to probabilistic logic about the degree of a

random variable [16]. Next, when Atanassov made another extension, he proposed the degree of falsity and studied the new structure set, which is called the intuitionistic fuzzy set [14]. Subsequentially, Smarandache made a new extension of the intuitionistic fuzzy set into the neutrosophic set by adding the degree of membership of indeterminacy. His work becomes a generalization of Lukasiewicz's and is associated with neutrosophic logic [7, 8, 9, 10, 11].

Previous mathematical contributions contributed to opening a wide window for mathematical researchers to reflect the mathematical concepts in classical set theory into new theories such as fuzzy set theory, intuitionistic fuzzy set theory, and neutrosophic set theory, respectively. Of course, the neutrosophic school was more prevalent than intuitionistic fuzzy and fuzzy schools because its founder, Smarandache, worked on writing in the various branches of mathematical sciences to contribute to its spread with other mathematical researchers, including the neutrosophic liner algebra, which we are concerned to talk about with regard to matrices and determinates only. For this purpose, we follow and investigate the literature on neutrosophic matrices and determinates, so we see a simple effect by pointing to the neutrosophic matrix and the product of two neutrosophic matrices in [21].

During the period between 2020 and 2023, we observe the first publication of working papers by researchers at the School of Neutrosophy, which is related to the algebra of matrices and determinants [3, 19, 20, 22]. The authors [19, 20] define the neutrosophic square matrices of of order n by $M = A + BI$, where A and B are two square neutrosophic matrices of order n. The determinant of matrix M is given by $det(M) = det(A) + I[det(A + B) - det(A)]$. The inverse of a neutrosophic matrix is defined as by $M^{-1} = A^{-1} + I[A + B)^{-1} - A^{-1}]$. They also study several properties of the inverse and determinants, with further details provided by the authors [22].

The authors [3] studied some results regarding the neutrosophic squares of complex matrices, using the same definitions of matrix, determinant, and inverse as in [19, 20, 22]. In [2], we explored algebraic structures of neutrosophic matrices based on the neutrosophic real number $a + bI$, taking different paths from [3, 19, 20, 22], and investigated various concepts and properties of neutrosophic matrices. Additionally, in [1], I established a connection with [2] through the concept of a neutrosophic ring of matrices. The present article aims to construct the neutrosophic determinant and explore its properties along the same path, rather than relying on [3, 19, 20, 22]. Lastly, I would like to highlight the earlier work of Dhar and Smarandache on a note about square neutrosophic fuzzy matrices, which can be found in [21].

2. Neutrosphic Determiantes

Neutrosophic algebra is a branch of the mathematical neutrosophic system, representing a new school of mathematics. Neutrosophic linear algebra, in turn, is a branch of neutrosophic algebra. In this section, we will compute the neutrosophic determinants, which rely on neutrosophic real numbers.

- **Definition 2.1.** [7] The standard form: : $a + b\,\mathrm{I}$ is called a neutrosophic number, where a, b are real coefficients, and $\mathrm{I} =$ indeterminacy. It follows that $0.\mathrm{I} = 0$ $and\ n\,\mathrm{I} = \mathrm{I}$ for all positive integers n.
- **Definition 2.2.** [3, 21, 22, 24] Let $M = A + BI$ represent a neutrosophic n square matrix. The determinant of M is defined as $detM = detA + I[det(A + B) - detA]$.
- **Definition 2.3.** [3, 21, 22, 24, 25] Let $\mathbb{R}$ be the set of real numbers. Then, $\mathrm{N}(\mathbb{R}) = \langle \mathbb{R} \cup \mathrm{I} \rangle = \{\mathrm{a} + \mathrm{bI}: \mathrm{a}, \mathrm{b} \in \mathbb{R}$ is a neutrosophic set, where $\mathrm{a} + \mathrm{bI}$ is a neutrosophic real number, and I is the indeterminate such that $0.I = 0, I^2 = I$.
- **Definition 2.4.** [27] Let R be any ring. The neutrosophic ring $\langle R \cup I \rangle$ is also a ring generated by R and I under the operations of R.

- **Definition 2.5.** [1] Let R be a nonempty set and the triple $(R,+,\bullet)$ be a ring, and consider the neutrosophic (NS):$R[I] = \{a + bI : a, b \in R\}$, then the neutrosophic algebra structure (NAS): $N(R) = \langle R[I],+,\bullet\rangle$ is called the neutrosophic associative ring, which is a generated by I and R under operations + " addition "and • " multiplications" respectively, if satisfies the axiomatic conditions of ring: NR1: For all x, y and $z \in N(R), N(R) = \langle R[I],+\rangle$ is a neutrosophic an abelian group under addition; NR2: For all x, y and $z \in N(R*), N(R*) = \langle R*[I],\bullet\rangle$ is a mathematical associative neutrosophic system under multiplications, that is, $N(R*) = \langle R*[I],\bullet\rangle$ is neutrosophic semi group and NR3: $x \bullet (y + z) = (x \bullet y) + (x \bullet z)$ and $(y + z) \bullet x = (y \bullet x) + (z \bullet x)$ " left and right distribution laws".

- **Definition 2.6.** [3, 21, 22, 24, 25] Let $\langle R \cup I\rangle$ be any neutrosophic ring. The collection of all $n \times n$ matrices with entries from $\langle R \cup I\rangle$ is called the neutrosophic matrix ring, i.e., $M_{n\times n} = \{M = (a_{ij}) \mid a_{ij} \in \langle R \cup I\rangle\}$. The operations are the usual matrix addition and matrix multiplication.

- **Definition2.7.**[2] Consider the neutrosophic matrix:
$M_{m\times n} = \{a_{ij} + b_{ij}I : a_{ij}, b_{ij} \in \mathbb{R}, 0I = 0 \;\&\; I^2 = I\}$
with m rows and n columns, then the scalar entry in the i^{th} neutrosophic row and in j^{th} neutrosophic column of neutrosophic matrix M is denoted by:
$a_{ij} + b_{ij}I$ *and is called the* $(i,j) - entry$ *of* M.

- **Definition 2.8.** [2] Consider the neutrosophic matrix set:

$$M_{m\times n} = \{a_{ij} + b_{ij}I : a_{ij}, b_{ij} \in R, 0I = 0 \;\&\; I^2 = I\} \quad (1).$$

with m rows and n columns, then the scalar entry in the i^{th} neutrosophic row and in j^{th} neutrosophic column of neutrosophic matrix M is denoted by $a_{ij} + b_{ij}I$ and is called the $(i,j) - entry$ of M. We wish to associate with this matrix a neutrosophic scalar that will in some sense the size or capacity of M and tell us whether or not M is non-singular of neutrosophic matrix. Consider the M is a square neutrosophic matrix of $n^{th} - order$:

$$M = \begin{bmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I & \cdots & a_{ij}+b_{ij}I & \cdots & a_{1n}+b_{1n}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I & \cdots & a_{ij}+b_{ij}I & \cdots & a_{2n}+b_{2n}I \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{i1}+b_{i1}I & a_{i2}+b_{i2}I & \ldots & a_{ij}+b_{ij}I & \ldots & a_{in}+b_{in}I \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}+b_{n1}I & a_{n2}+b_{n2}I & \cdots & a_{nj}+b_{nj}I & \cdots & a_{nn}+b_{nn}I \end{bmatrix} \quad (2).$$

$$\text{Then: } a_k + b_kI = \begin{bmatrix} a_{1k}+b_{1k}I \\ a_{2k}+b_{2k}I \\ a_{3k}+b_{3k}I \\ \vdots \\ a_{nk}+b_{nk}I \end{bmatrix} \quad (3).$$

which identify with k^{th} neutrosophic column vector of M,consider the i^{th} neutrosophic rowvector of M which given by: $x_i + x'_iI = [a_{i1} + b_{i1}I, a_{i2} + b_{i2}I, \ldots, a_{ik} + b_{ik}I, \ldots, a_{in} + b_{in}I]$ (4), where $i = 1,2,3,\ldots,n$; $k = 1,2,3,\ldots,n$.

- **Definition 2.9.** [8] Let U be a universe of discourse, and let us consider $A \subset U$.
A neutrosophic set A is defined as an object having the form:

$A = \{(x, T_A(x)), (x, I_A(x)), (x, F_A(x))\}$, where $T_A(x), I_A(x)$ and $F_A(x) \in [0,1],$) represent the degree of membership, degree of indeterminacy, and degree of non-membership, respectively, of each element and the sum:

$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 1$, for all $x \in U$.

Now, we introduce a new definition related to neutrosophic determinants, which is analogous to classical linear algebra.

- **Definition 2.10.** Consider the square neutrosophic matrix of $\mathrm{n^{th}} -$ order, then delete the $\mathrm{i^{th}}$ row and the $\mathrm{k^{th}}$ column from it yields a reduction neutrosophic matrix or neutrosophic submatrix of the $(\mathrm{n}-1)^{\mathrm{th}} -$ order. This submatrix corresponds to the neutrosophic element $a_{ik} + b_{ik}I$ and is denoted by:

$$M_{ik} = \begin{bmatrix} a_{11} + b_{11}I & \cdots & a_{1(k-1)} + b_{1(k-1)}I & a_{1(k+1)} + b_{1(k+1)1}I & \cdots & a_{1n} + b_{1n}I \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(i-1)1} + b_{(i-1)1}I & \ldots & a_{(i-1)(k-1)} + b_{(i-1)(k-1)}I & a_{(i-1)(k+1)} + b_{(i-1)(k+1)}I & \ldots & a_{(i-1)n} + b_{(i-1)n}I \\ a_{(i+1)1} + b_{(i-1)}I & \cdots & a_{(i+1)(k-1)} + b_{(k-1)}I & a_{(i+1)(k+1)} + b_{(k+1)}I & \cdots & a_{(i+1)n} + b_nI \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} + b_{n1}I & \cdots & a_{n(k-1)} + b_{n(k-1)}I & a_{n(k+1)} + b_{n(k+1)}I & \cdots & a_{nn} + b_{nn}I \end{bmatrix}$$

By applying the same technique, we can delate both the $\mathrm{i^{th}}$ and the $\mathrm{j^{th}}$ row, as well as the $\mathrm{k^{th}}$ and the $\mathrm{l^{th}}$ column, to obtain a neutrosophic submatrix of the order $(\mathrm{n}-\mathrm{n})^{\mathrm{th}}$, denoted as $\mathrm{M_{(ik)(jl)}}$.

- **Definition 2.11**. Consider the neutrosophic matrix M of the $1^{\mathrm{st}} -$ order of the form $[\mathrm{a_{11}} + \mathrm{b_{11}I}]$. The neutrosophic determinant is defined as follows:

$\mathrm{N(det[a_{11} + b_{11}I]) = a_{11} + b_{11}I}$, for instance, $N(det[2 + 3I]) = 2 + 3I$.

- **Definition 2.12**. Consider the neutrosophic matrix M of the $2^{\mathrm{nd}} -$ order of the form:

$$\mathrm{M} = \begin{bmatrix} \mathrm{a_{11} + b_{11}I} & \mathrm{a_{12} + b_{12}I} \\ \mathrm{a_{21} + b_{21}I} & \mathrm{a_{22} + b_{22}I} \end{bmatrix} \quad (1).$$

Then the neutrosophic determinant is defined as follows:

$$\mathrm{N(det(M))} = \begin{vmatrix} \mathrm{a_{11} + b_{11}I} & \mathrm{a_{12} + b_{12}I} \\ \mathrm{a_{21} + b_{21}I} & \mathrm{a_{22} + b_{22}I} \end{vmatrix}$$
$$= \mathrm{(a_{11} + b_{11}I)(a_{22} + b_{22}I) - (a_{21} + b_{21}I)(a_{12} + b_{12}I)} \quad (2).$$

For instance, if $\mathrm{M} = \begin{bmatrix} 2+\mathrm{I} & 3+4\mathrm{I} \\ 5-\mathrm{I} & 1+2\mathrm{I} \end{bmatrix}$, then $\mathrm{N(det(M))} = \begin{vmatrix} 2+\mathrm{I} & 3+4\mathrm{I} \\ 5-\mathrm{I} & 1+2\mathrm{I} \end{vmatrix}$.

- **Definition2.13.** Consider the neutrosophic matrix M of the $\mathrm{n^{th}} -$ order

$(\mathrm{n} > 2, \mathrm{n} \in \mathbb{Z}^+)$, then the neutrosophic determinant is defined as follows:

$$N(det\,(M) = \begin{bmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I & \cdots & a_{ij} + b_{ij}I & \cdots & a_{1n} + b_{1n}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I & \cdots & a_{ij} + b_{ij}I & \cdots & a_{2n} + b_{2n}I \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} + b_{i1}I & a_{i2} + b_{i2}I & \ldots & a_{ij} + b_{ij}I & \ldots & a_{in} + b_{in}I \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} + b_{n1}I & a_{n2} + b_{n2}I & \cdots & a_{nj} + b_{nj}I & \cdots & a_{nn} + b_{nn}I \end{bmatrix}$$
$$= \sum_{k=1}^{n} (a_{1k} + b_{1k}I)\,(-1)^{1+k}\, N(det(M_{1k}) \quad (1).$$

Where $\mathrm{M_{1k}}$ is the neutrosophic submatrix related to the neutrosophic element $(\mathrm{a_{1k} + b_{1k}I})$, $\mathrm{k} = 1,2,3, \ldots, \mathrm{n}$. The neutrosophic Determinate of neutrosophic submatrix $\mathrm{M_{ik}}$ of neutrosophic matrix in Definition 2.5, denoted by $\mathrm{N(det\,(M_{ik})}$ is called the neutrosophic minor of neutrosophic entry $(\mathrm{a_{ik} + b_{ik}I})$, and the neutrosophic number

$$C_{ik} = (-1)^{i+k}\, N(det(M_{ik}) \qquad (2).$$

Substitute equation (2) with equation (1) to obtain the formula:

$$N(det(M) = \sum_{k=1}^{n}(a_{1k} + b_{1k}I)\, C_{ik} \qquad (3).$$

Formula (3) and (1) are referred to as the expansion of determinants by their first row. We compute the neutrosophic determinate with " +" or "– " sign relating to C_{ik} with the following sign matrix:

$$sign = \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Example 2.1. Consider the neutrosophic matrix M of the $4^{\text{th}} -$ order, which is given as follows:

$$M = \begin{bmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I & a_{13} + b_{13}I & a_{14} + b_{14}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I & a_{23} + b_{23}I & a_{24} + b_{24}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I & a_{33} + b_{33}I & a_{34} + b_{34}I \\ a_{41} + b_{41}I & a_{42} + b_{42}I & a_{43} + b_{43}I & a_{44} + b_{44}I \end{bmatrix}$$

(1). For instance,

$$= \begin{bmatrix} 1 + I & 3 + 2I & 5 + 4I & 1 - I \\ 2 - I & 4 - 6I & 2 + 2I & 10 - I \\ 0 + 4I & 1 + 4I & 3 + 3I & 0 + 5I \\ 1 - 3I & 1 + 2I & 9 - I & 0 + 2I \end{bmatrix} \qquad (2).$$

To compute the N(det (M)).$N(det\,(M) \;= \sum_{k=1}^{n}(a_{1k} + b_{1k}I)\,(-1)^{1+k}\, N(det(M_{1k})$ (3).

$$= \sum_{k=1}^{4}(a_{1k} + b_{1k}I)\,(-1)^{1+k}\, N(det(M_{1k}). \qquad (4).$$

$$= \begin{pmatrix} (1 + I)(-1)^2\, N(det(M_{11}) \\ + \\ (3 + 2I)(-1)^3\, N(det(M_{12}) \\ + \\ (5 + 4I)(-1)^4\, N(det(M_{13}) \\ + \\ (1 - I)(-1)^5\, N(det(M_{14}) \end{pmatrix} \qquad (5).$$

$$= \begin{pmatrix} (1 + I)\,(1) \begin{vmatrix} 4 - 6I & 2 + 2I & 10 - I \\ 1 + 4I & 3 + 3I & 0 + 5I \\ 1 + 2I & 9 - I & 0 + 2I \end{vmatrix} \\ + \\ (3 + 2I)(-1) \begin{vmatrix} 2 - I & 2 + 2I & 10 - I \\ 0 + 4I & 3 + 3I & 0 + 5I \\ 1 - 3I & 9 - I & 0 + 2I \end{vmatrix} \\ + \\ (5 + 4I)(1) \begin{vmatrix} 2 - I & 4 - 6I & 10 - I \\ 0 + 4I & 1 + 4I & 0 + 5I \\ 1 - 3I & 1 + 2I & 0 + 2I \end{vmatrix} \\ + \\ (1 - I)(-1) \begin{vmatrix} 2 - I & 4 - 6I & 2 + 2I \\ 0 + 4I & 1 + 4I & 3 + 3I \\ 1 - 3I & 1 + 2I & 9 - I \end{vmatrix} \end{pmatrix} \qquad (6).$$

$$
=\left(\begin{array}{c}
(1+I)\,(1)\left\{\begin{array}{c}
(1)(4-6I)\left\{\begin{vmatrix}3+3I & 0+5I\\ 9-I & 0+2I\end{vmatrix}\right\}\\
(-1)(2+2I)\left\{\begin{vmatrix}1+4I & 0+5I\\ 1+2I & 0+2I\end{vmatrix}\right\}\\
+(1)(10-I)\left\{\begin{vmatrix}1+4I & 3+3I\\ 1+2I & 9-I\end{vmatrix}\right\}
\end{array}\right\}\\
+\\
(3+2I)(-1)\left\{\begin{array}{c}
(1)(2-I)\left\{\begin{vmatrix}3+3I & 0+5I\\ 9-I & 0+2I\end{vmatrix}\right\}\\
(-1)(2+2I)\left\{\begin{vmatrix}0+4I & 0+5I\\ 1-3I & 0+2I\end{vmatrix}\right\}\\
+(1)(10-I)\left\{\begin{vmatrix}0+4I & 3+3I\\ 1-3I & 9-I\end{vmatrix}\right\}
\end{array}\right\}\\
+\\
(5+4I)(1)\left\{\begin{array}{c}
(1)(2-I)\left\{\begin{vmatrix}1+4I & 0+5I\\ 1+2I & 0+2I\end{vmatrix}\right\}\\
(-1)(4-6I)\left\{\begin{vmatrix}0+4I & 0+5I\\ 1-3I & 0+2I\end{vmatrix}\right\}\\
+(1)(10-I)\left\{\begin{vmatrix}0+4I & 1+4I\\ 1-3I & 1+2I\end{vmatrix}\right\}
\end{array}\right\}\\
+\\
(1-I)(-1)\left\{\begin{array}{c}
(1)(2-I)\left\{\begin{vmatrix}1+4I & 3+3I\\ 1+2I & 9-I\end{vmatrix}\right\}\\
(-1)(4-6I)\left\{\begin{vmatrix}0+4I & 3+3I\\ 1-3I & 9-I\end{vmatrix}\right\}\\
+(1)(2+2I)\left\{\begin{vmatrix}0+4I & 1+4I\\ 1-3I & 1+2I\end{vmatrix}\right\}
\end{array}\right\}
\end{array}\right) \quad (7).
$$

$$
=\left(\begin{array}{c}
(1+I)\,(1)\left\{\begin{array}{c}
(1)(4-6I)\left\{\big((3+3I)(0+2I)\big)-\big((9-I)(0+5I)\big)\right\}\\
(-1)(2+2I)\left\{\big((1+4I)(0+2I)\big)-\big((1+2I)(0+5I)\big)\right\}\\
+(1)(10-I)\left\{\big((1+4I)(9-I)\big)-\big((1+2I)(3+3I)\big)\right\}
\end{array}\right\}\\
+\\
(3+2I)(-1)\left\{\begin{array}{c}
(1)(2-I)\left\{\big((3+3I)(0+2I)\big)-\big((9-I)(0+5I)\big)\right\}\\
(-1)(2+2I)\left\{\big((0+4I)(0+2I)\big)-\big((1-3I)(0+5I)\big)\right\}\\
+(1)(10-I)\left\{\big((0+4I)(9-I)\big)-\big((1-3I)(3+3I)\big)\right\}
\end{array}\right\}\\
+\\
(5+4I)(1)\left\{\begin{array}{c}
(1)(2-I)\left\{\big((1+4I)(0+2I)\big)-\big((1+2I)(0+5I)\big)\right\}\\
(-1)(4-6I)\left\{\big((0+4I)(0+2I)\big)-\big((1-3I)(0+5I)\big)\right\}\\
+(1)(10-I)\left\{\big((0+4I)(1+2I)\big)-\big((1-3I)(1+4I)\big)\right\}
\end{array}\right\}\\
+\\
(1-I)(-1)\left\{\begin{array}{c}
(1)(2-I)\left\{\big((1+4I)(9-I)\big)-\big((1+2I)(3+3I)\big)\right\}\\
(-1)(4-6I)\left\{\big((0+4I)(9-I)\big)-\big((1-3I)(3+3I)\big)\right\}\\
+(1)(2+2I)\left\{\big((0+4I)(1+2I)\big)-\big((1-3I)(1+4I)\big)\right\}
\end{array}\right\}
\end{array}\right) \quad (8).
$$

$$
=\left(\begin{array}{c}
(1+I)\,(1)\left\{\begin{array}{c}
(1)(4-6I)\,\{0-28I\}\\
(-1)(2+2I)\{0-5I\}\\
+(1)(10-I)\{6+16I\}
\end{array}\right\}\\
+\\
(3+2I)(-1)\left\{\begin{array}{c}
(1)(2-I)\,\{0-28I\}\\
(-1)(2+2I)\{0+18I\}\\
+(1)(10-I)\{-3+47I\}
\end{array}\right\}\\
+\\
(5+4I)(1)\left\{\begin{array}{c}
(1)(2-I)\,\{0-5I\}\\
(-1)(4-6I)\{0+18I\}\\
+(1)(10-I)\{-1+23I\}
\end{array}\right\}\\
+\\
(1-I)(-1)\left\{\begin{array}{c}
(1)(2-I)\,\{6+16I\}\\
(-1)(4-6I)\{-3+47I\}\\
+(1)(2+2I)\{-1+I\}
\end{array}\right\}
\end{array}\right) \quad (9).
$$

$$= \begin{pmatrix} (1+I)\,(1)\begin{Bmatrix} 0+56I \\ +0+20I \\ +60+138I \end{Bmatrix} \\ + \\ (3+2I)(-1)\begin{Bmatrix} 0+84I \\ +0-72I \\ -30+97I \end{Bmatrix} \\ + \\ (5+4I)(1)\begin{Bmatrix} 0-5I \\ 0+36I \\ -10+101I \end{Bmatrix} \\ + \\ (1-I)(-1)\begin{Bmatrix} 12+10I \\ -12-76I \\ -2+2I \end{Bmatrix} \end{pmatrix} \quad (10).$$

$$= \begin{pmatrix} (1+I)\,(60+214I) \\ + \\ (3+2I)(30-109I) \\ + \\ (5+4I)(-10+132I) \\ + \\ (1-I)(-12+45I) \end{pmatrix} \quad (11).$$

$$= \begin{pmatrix} 60+488I \\ + \\ 90-485I \\ + \\ -50+1148I \\ + \\ -12+12I \end{pmatrix} \quad (12).$$

$$= 88 + 1299I. \quad (13).$$

3. Some Properties of Neutrosphic Determinates

In this section, we investigated the main result related to neutrosophic determinants that correspond to the classical linear algebra theory.

Theorem3.1. Let $\mathrm{M} = [\mathrm{a_{ij}} + \mathrm{b_{ij}}\mathrm{I}]_{\mathrm{n\times n}}$ be a neutrosophic square matrix of $n^{th}-$order, which contains a neutrosophic row of neutrosophic zeros, then the $N(det\ (M) = 0 + 0I$.

Proof. Since N(det (M) is given by:

$$\begin{aligned} N(det\ (M) \ &= \textstyle\sum_{k=1}^{n}(a_{1k} + b_{1k}I)\,(-1)^{1+k}\, N(det(M_{1k}) \\ &= N(det(M)) \\ &= \textstyle\sum_{k=1}^{n}(0_{1k} + 0_{1k}I)\,(-1)^{1+k}\, N(det(M_{1k}) \qquad (1). \end{aligned}$$

where M_{1k} represents the neutrosophic submatrix associated with the neutrosophic element $(0_{1k} + 0_{1k}\mathrm{I})$. So, the product of a cofactor of neutrosophic$(-1)^{1+k}\, N(det(M_{1k})$ of neutrosophic entry $(0_{1k} + 0_{1k}I)$ from each row neutrosophic matrix M, every signed elementary product contains a neutrosophic factor from the neutrosophic of zeros so that has value neutrosophic zero, hence $N(det(M)) = 0 + 0I$■.

Observation: The neutrosophic determinant can be expanded using any neutrosophic row or column that contains more zeros.

Example 3.1. Consider the neutrosophic matrix:

$$M = \begin{bmatrix} 2I & 3I & 4I \\ 2+4I & 3-2I & 1-2I \\ 0 & 0 & 0 \end{bmatrix}$$, then it is clear that,

$$N(det\,(M) \underbrace{=}_{by\, r-3} 0.\begin{vmatrix} 3I & 4I \\ 3-2I & 1-2I \end{vmatrix} - 0.\begin{vmatrix} 2I & 4I \\ 2+4I & 1-2I \end{vmatrix} + 0.\begin{vmatrix} 2I & 3I \\ 2+4I & 3-2I \end{vmatrix} = 0.$$

Theorem3.2. Let $M = [a_{ij} + b_{ij}I]_{n\times n}$ be a neutrosophic square matrix of $n^{th} - order$. If M' is a neutrosophic matrix that is obtained when the

i^{th} $(i = 1,2,3, \dots, n)$ neutrosophic row vector is multiplied by neutrosophic scalar $x = x_1 + x_2I$, then: $N(det(M') = N(det(xM) \neq x^n . N(det(M))$.

Proof. By example.

Example 3.2. Consider the neutrosophic square matrix:$M = \begin{bmatrix} 1+I & 2-I \\ 3+I & 6-I \end{bmatrix}$, then:

$$N(det(M)) = \begin{vmatrix} 1+I & 2-I \\ 3+I & 6-I \end{vmatrix} = \big((1+I).(6-I) - \big((3+I).(2-I)\big)\big)$$

$$= \big(6 + 4I - (6 - 2I)\big) = \big(6 + 4I - (6 - 2I)\big) = 6I.$$

$$Take\ x = 2 + 2I, then\ x.M = (2+2I) \begin{bmatrix} 1+I & 2-I \\ 3+I & 6-I \end{bmatrix} = \begin{bmatrix} 2+6I & 0 \\ 6+10I & 12+8I \end{bmatrix},$$

$$N(det(x.M)) = \begin{vmatrix} 2+6I & 0 \\ 6+10I & 12+8I \end{vmatrix} = \big((2+6I).(12+8I) - (0)\big) = (24 + 136I).$$

We note that. $x^2 = (2+2I)^2 = 4 + 12I\ and\ (4 + 12I).6I = 96I$, therefore $N(det(M') = N(det(xM) \neq x^n . N(det(M))$.

Corollary 3.1. Let $M = [a_{ij} + b_{ij}I]_{n\times n}$ be a neutrosophic square matrix of $n^{th} - order$. If M' is a neutrosophic matrix that is obtained when the

i^{th} $(i = 1,2,3, \dots, n)$ neutrosophic row vector is multiplied by neutrosophic scalar $x = x_1\ or\ x = x_2I$, then: $N(det(M') = N(det(xM) = x^n . N(det(M))$.

Proof. Suppose that

$$N(det\,(M') = \begin{bmatrix} x.(a_{11}+b_{11}I) & x.(a_{12}+b_{12}I) & \cdots & x.\big(a_{ij}+b_{ij}I\big) & \cdots & x.(a_{1n}+b_{1n}I) \\ x.(a_{21}+b_{21}I) & x.(a_{22}+b_{22}I) & \cdots & x.\big(a_{ij}+b_{ij}I\big) & \cdots & x.(a_{2n}+b_{2n}I) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x.(a_{i1}+b_{i1}I) & x.(a_{i2}+b_{i2}I) & \dots & x.\big(a_{ij}+b_{ij}I\big) & \dots & x.(a_{in}+b_{in}I) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x.(a_{n1}+b_{n1}I) & x.(a_{n2}+b_{n2}I) & \cdots & x.\big(a_{nj}+b_{nj}I\big) & \cdots & x.(a_{nn}+b_{nn}I) \end{bmatrix} \quad (1).$$

$$N(det(M') = \textstyle\sum_{k=1}^{n} \big(x.(a_{1k} + b_{1k}I)\big)\, C_{1k} \quad (2).$$

$$= \begin{pmatrix} \big(x.(a_{11}+b_{11}I)\big)\, C_{11} \\ + \\ \vdots \\ + \\ \big(x.(a_{1h}+b_{1h}I)\big)\, C_{1h} \\ + \\ \big(x.(a_{1h+1}+b_{1h+1}I)\big)\, C_{1h+1} \\ + \\ \vdots \\ \big(x.(a_{1n}+b_{1n}I)\big)\, C_{1n} \end{pmatrix} \quad (3).$$

$$= \begin{pmatrix} x.\left((a_{11} + b_{11}I)C_{11}\right) \\ + \\ \vdots \\ + \\ x.\left((a_{1h} + b_{1h}I)C_{1h}\right) \\ + \\ x.\left((a_{1h+1} + b_{1h+1}I)C_{1h+1}\right) \\ + \\ \vdots \\ x.\left((a_{1n} + b_{1n}I)C_{1n}\right) \end{pmatrix} \quad (4).$$

$$= x^n. \begin{pmatrix} \left((a_{11} + b_{11}I)C_{11}\right) \\ + \\ \vdots \\ + \\ \left((a_{1h} + b_{1h}I)C_{1h}\right) \\ + \\ \left((a_{1h+1} + b_{1h+1}I)C_{1h+1}\right) \\ + \\ \vdots \\ \left((a_{1n} + b_{1n}I)C_{1n}\right) \end{pmatrix} \quad (5).$$

$$= x^n.\sum_{k=1}^{n}(a_{1k} + b_{1k}I)\, C_{ik} = x^n.\, N(det(M) \quad (6).$$

As we observe, each term in the expansion of the neutrosophic determinant contains only one neutrosophic scalar from each neutrosophic row of the neutrosophic matrix M■.

Example 3.3. Let $M = \begin{bmatrix} 3I & I \\ 2I & 2I \end{bmatrix}$ be a neutrosophic square matrix.

Consider $M' = 5I \begin{bmatrix} 3I & I \\ 2I & 2I \end{bmatrix} = \begin{bmatrix} 15 & 5I \\ 10I & 10I \end{bmatrix}$, to compute $N(det(M)) = 4I$ and the $N(det(M')) = N(det(5I)M) = 100I$, so, result satisfies with relationship of Corollary 3.1. that is, $N(det(M') = N(det(5IM) = 5^2.4I = x^2N(det(M))$.

Theorem3.3. Let $M = [a_{ij} + b_{ij}I]_{n\times n}$ be a neutrosophic square matrix of $n^{th} - order$. If M' is a neutrosophic matrix that is obtained from M by interchanging any two neutrosophic rows (or columns) of M, then $N(det(M')) = (-1)N(det(M))$.

Proof. Assume that

4.

$$N(det\,(M) = \begin{bmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I & \cdots & a_{ij} + b_{ij}I & \cdots & a_{1n} + b_{1n}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I & \cdots & a_{ij} + b_{ij}I & \cdots & a_{2n} + b_{2n}I \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} + b_{i1}I & a_{i2} + b_{i2}I & \dots & a_{ij} + b_{ij}I & \dots & a_{in} + b_{in}I \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} + b_{n1}I & a_{n2} + b_{n2}I & \cdots & a_{nj} + b_{nj}I & \cdots & a_{nn} + b_{nn}I \end{bmatrix} \quad (1).$$

$$= \sum_{k=1}^{n}(a_{1k} + b_{1k}I)\,(-1)^{1+k}\, N(det(M_{1k}) \quad (2).$$

Now, if we interchange the arbitrary neutrosophic rows i^{th} and l^{th} of M respectively, where $x_i + x'_iI = [a_{i1} + b_{i1}I, a_{i2} + b_{i2}I, \dots, a_{ik} + b_{ik}I, \dots, a_{in} + b_{in}I]$ *(3).* And

$y_l + x'_lI = [a_{l1} + b_{l1}I, a_{l2} + b_{l2}I, \dots, a_{lk} + b_{lk}I, \dots, a_{ln} + b_{ln}I]$ *(4).*

If $1 \le i < l \le n$, *then* $1 \le l - i \le n$, hence the i^{th} neutrosophic row is transported into the l^{th} position in $(l - i) - steps$; therefore, the previous of $n - row$ l^{th} is transported into the position in

$(l-i-1)-steps,$ so the formula of sign in (1) becomes: $(-1)^{(l-i)+(l-i-1)} = (-1)^{2(l-i)-1} = -1$, since the power is odd number. Hence $N(det(M')) = (-1)N(det(M)$■.

Example 3.4. Consider the neutrosophic matrix: $M = \begin{bmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{bmatrix}$, then:

$N(det(M)) = \begin{bmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{bmatrix} = 2+8I$. If we make interchange row-1by row-2to get $N(det(M')) = \begin{bmatrix} 2+2I & 2+4I \\ 2+3I & 1+4I \end{bmatrix} = (2+18I)-(4+26I) = -2-8I$.We note that $N(det(M')) = (-1)N(det(M)$.

Theorem3.4. Let $M = [a_{ij}+b_{ij}I]_{n\times n}$ be a neutrosophic square matrix of $n^{th}-order$. If M has two equal rows then $N(det(M) = 0+0I$.

Proof. Suppose that $M = [a_{ij}+b_{ij}I]_{n\times n}$ has two identical rows, the interchanging two rows yield the identical Neutrosophic matrix M; therefore,

$N(det(M)) = (-1)N(det(M)$ by theorem 3.3, hence $N(det(M) + N(det(M) = 0$ *iff* $N(det(M)) = 0$ *iff* $N(det(M)) = 0+0I$■.

Theorem3.5. Let $M = [a_{ij}+b_{ij}I]_{n\times n}$ and $N = [a'_{ij}+b'_{ij}I]_{n\times n}$ be two neutrosophic square matrices of the same $\mathrm{n^{th}-order}$. Then:

$$N(det(M+N)) \neq N(det(M)) + N(det(N)).$$

This theorem does not work like its counterparts in classical linear algebra, as demonstrated by the following example.

Example 3.4. *Let* $M = \begin{bmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{bmatrix}$ and $N = \begin{bmatrix} 1+I & 3+2I \\ 2-I & 1-I \end{bmatrix}$ be two neutrosophic matrices, then:

$N(det(M)) = \begin{vmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{vmatrix} = \big((4+26I)-(2+18I)\big) = 2+8I$. Also,

$N(det(N)) = \begin{vmatrix} 1+I & 3+2I \\ 2-I & 1-I \end{vmatrix} = \big((1-I)-(6-I)\big) = -5.$ *Now,*

$$N(det(M)) + N(det(N)) = (2+8I) + (-5) = -3+8I.$$

$$M+N = \begin{bmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{bmatrix} + \begin{bmatrix} 1+I & 3+2I \\ 2-I & 1-I \end{bmatrix} = \begin{bmatrix} 3+4I & 4+6I \\ 4+I & 3+3I \end{bmatrix}$$

$$N(det(M+N)) = \begin{vmatrix} 3+4I & 4+6I \\ 4+I & 3+3I \end{vmatrix} = \big((3+4I).(3+3I)\big) - \big((4+I).(4+6I)\big)$$
$$= \big((9+33I)-(16+44I)\big)$$
$$= -7-11I.$$

As we see, $N(det(M+N)) \neq N(det(M)) + N(det(N))$.

Example 3.5. Consider the two neutrosophic matrices:

$M = \begin{bmatrix} 3I & I \\ 2I & 2I \end{bmatrix}$ and $N = \begin{bmatrix} 15 & 5I \\ 10I & 10I \end{bmatrix}$.

the $\mathrm{N(det(M)) = 4I}$ and $\mathrm{N(det(N)) = 100I}$, Now, $\mathrm{N(det(M)) + N(det(N)) = 104I}$.

$$\mathrm{M+N} = \begin{bmatrix} 3I & I \\ 2I & 2I \end{bmatrix} + \begin{bmatrix} 15 & 5I \\ 10I & 10I \end{bmatrix} = \begin{bmatrix} 15+3I & 6I \\ 12I & 12I \end{bmatrix}$$

$$\mathrm{N(det(M+N))} = \begin{vmatrix} 15+3I & 6I \\ 12I & 12I \end{vmatrix} = \Big((15+3I).(12I) - \big((12I).(6I)\big)\Big)$$
$$= \big(216I - (72I)\big) = 144I. \text{ As we see,}$$
$$\mathrm{N(det(M+N)) \neq N(det(M)) + N(det(N))}.$$

Theorem3.5. Let $\mathrm{M = [a_{ij}+b_{ij}I]_{n\times n}}$ and $\mathrm{N = [c_{ij}+d_{ij}I]_{n\times n}}$ be two neutrosophic square matrix of the $\mathrm{n^{th}-order}$. Then: $\mathrm{N(det(M.N)) = N(det(M)).N(det(N))}$.

Example 3.6. Let $\mathrm{M} = \begin{bmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{bmatrix}$ and $\mathrm{N} = \begin{bmatrix} 1+I & 3+2I \\ 2-I & 1-I \end{bmatrix}$ be two neutrosophic matrices, then:

$\mathrm{N(det(M))} = \begin{vmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{vmatrix} = \big((4+26I)-(2+18I)\big) = 2+8I$. Also,

$$N(\det(N)) = \begin{vmatrix} 1+I & 3+2I \\ 2-I & 1-I \end{vmatrix} = \big((1-I)-(6-I)\big) = -5. \text{ Hence,}$$
$$N(\det(M)).N(\det(N)) = (2+8I).(-5) = -10-40I.$$
$$MN = \begin{bmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{bmatrix}.\begin{bmatrix} 1+I & 3+2I \\ 2-I & 1-I \end{bmatrix}$$
$$= \begin{bmatrix} (2+3I).(1+I)+(1+4I)(2-I) & (2+3I).(3+2I)+\big((1+4I).(1-I)\big) \\ (2+2I).(1+I)+(2+4I).(2-I) & (2+2I).(3+2I)+(2+4I).(1-I) \end{bmatrix}$$
$$= \begin{bmatrix} (2+8I)+(2+3I) & (6+19I)+(1-I) \\ (2+6I)+(4+2I) & (6+14I)+(2-2I) \end{bmatrix}$$
5. $= \begin{bmatrix} 4+11I & 7+18I \\ 6+8I & 8+12I \end{bmatrix}$. Now,
$$N(\det(MN)) = \begin{vmatrix} 4+11I & 7+18I \\ 6+8I & 8+12I \end{vmatrix} = \big((4+11I).(8+12I)\big) - \big((6+8I).(7+18I)\big)$$
$$= \big((32+268I)-(42+308I)\big) = -10-40I.$$

Theorem3.6. Let $\boldsymbol{M} = [\boldsymbol{a_{ij}} + \boldsymbol{b_{ij}I}]_{\boldsymbol{n\times n}}$ be a neutrosophic square matrix of the $\boldsymbol{n^{th}-order}$, then $\boldsymbol{N(det(M^t)) = N(det(M))}$.

Example 3.7. Let $M = \begin{bmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{bmatrix}$ be a neutrosophic matrix of the $2^{nd}-order$, then

$$N(det(M)) = \begin{vmatrix} 2+3I & 1+4I \\ 2+2I & 2+4I \end{vmatrix} = \big((4+26I)-(2+18I)\big) = 2+8I. \text{ Also,}$$

$$M^t = \begin{bmatrix} 2+3I & 2+2I \\ 1+4I & 2+4I \end{bmatrix}, \text{ and } N(det(M^t)) = \begin{vmatrix} 2+3I & 2+2I \\ 1+4I & 2+4I \end{vmatrix}$$
$$= \big((2+3I).(2+4I)\big) - \big((1+4I).(2+2I)\big)$$
$$= \big((4+26I)-(2+18I)\big) = 2+8I.$$

We deduce that $N(det(M^t)) = N(det(M))$.

4. Conclusion

This paper introduces a novel algebraic structure called neutrosophic determinants, which are based on the neutrosophic real number $\boldsymbol{a+bI}$. The paper also presents theorems and examples that highlight the distinctions between classical linear algebra and neutrosophic linear algebra. Furthermore, this work complements previous research on the algebraic structure of neutrosophic matrices. In future studies, we will study other concepts related to classical linear algebra, such as neutrosophic matrix systems of equations, neutrosophic inverse of matrices, neutrosophic vector spaces, linear independence, and more.

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