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On the Connection Between Lovelock Gravity and the Poincaré-Hopf Index

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Abstract

This study explores a novel conceptual link between the dynamics of Lovelock gravity and the topological structure of spacetime. We introduce a theoretical framework in which the gravitational action—particularly in Lovelock theories—is connected to the sum of topological indices of a vector field defined on the spacetime manifold, consistent with the Poincaré-Hopf theorem. Specifically, we propose that the value of the pth-order Lovelock action, recognized as a topological invariant in $2p$ dimensions, corresponds to the sum of indices associated with topological defects such as black hole horizons or spacetime singularities. Although the explicit construction of this vector field is left for future work, we present a foundational formulation and demonstrate its implications using the Schwarzschild solution, where the index sum reproduces the known Euler characteristic. This approach offers a topologically grounded perspective on gravitational dynamics and suggests new directions for understanding the geometric and physical properties of spacetime.

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Lovelock gravity, spacetime, Poincaré-Hopf theorem, black hole horizon, spacetime singularities, Schwarzschild solution, Euler characteristic.

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1. INTRODUCTION

The relationship between the geometry of spacetime and the matter and energy within it forms the basis of Einstein's theory of general relativity. A deeper question, however, is the extent to which the topology of spacetime, its fundamental connectedness and structure, governs the laws of physics. Lovelock's theory of gravity provides the most natural generalization of Einstein's theory to higher dimensions [1]. Its action is constructed from higher-order curvature invariants, yet remarkably, it yields second-order equations of motion, avoiding the instabilities that plague other higher-derivative theories [2].

A key feature of Lovelock gravity is its intimate connection with topology. In specific dimensions, the Lovelock Lagrangian's are equivalent to Euler densities, meaning their integral over a compact manifold yields the Euler characteristic, a pure topological invariant [3]. For instance, in four dimensions, the second-order Lovelock (Gauss-Bonnet) term is a total derivative whose integral is proportional to the Euler characteristic $\chi(\mathcal{M})$.

Independently, the Poincaré-Hopf theorem provides a powerful tool in differential topology, linking the local behavior of a vector field to the global topology of the manifold on which it is defined [4]. The theorem states that for a suitable vector field $\vec{v}_{grav}$ on a compact, orientable manifold $\mathcal{M}$, the sum of the indices of its isolated zeros is equal to the Euler characteristic of $\mathcal{M}$ $\sum_{x_i \in \text{zeros}} \text{Ind}_{x_i}(\vec{v}_{grav}) = \chi(\mathcal{M})$ where $\vec{v}_{grav}$ is any smooth vector tanget field.

This paper seeks to bridge these two powerful ideas. We propose that the connection between Lovelock gravity and topology can be made more concrete by identifying the Euler characteristic not just as an abstract number, but as a sum of indices of a physically meaningful vector field. The zeros of this field would correspond to topological defects in the spacetime, such as black hole horizons or singularities.

Our central thesis is this: *The integral of the Lovelock action over a manifold is proportional to the sum of topo-*

logical indices of a characteristic vector field defined on that manifold. We will formalize this connection and explore its consequences. While the rigorous construction of this vector field remains a formidable challenge, we will postulate its necessary properties and apply the framework to the well-understood case of a Schwarzschild black hole, demonstrating the consistency and potential of this approach.

This relationship has been leveraged in the context of gravity to associate the number and type of singularities with global topological quantities [4].

Recent studies such as that by Misba-Afrin al. [5] have applied the Hopf index in the thermodynamic phase space of black holes. They interpret black hole solutions as topological defects whose indices sum to a conserved topological number. This provides a new classification scheme for black holes using the winding number of a free energy vector field.

1.1. Connecting Lovelock Gravity and the Poincaré-Hopf Index

Lovelock gravity represents a natural generalization of Einstein's theory to higher dimensions, yielding second-order field equations while being free of ghosts. The theory is constructed from dimensionally extended Euler densities. For a D-dimensional manifold, the p-th Lovelock Lagrangian, $\mathcal{L}_{(p)}$, is a scalar invariant built from the Riemann tensor, defined as:

$$\mathcal{L}_{(p)} = \frac{1}{2^p}\delta^{a_1 b_1 \ldots a_p b_p}_{c_1 d_1 \ldots c_p d_p} R^{c_1 d_1}_{a_1 b_1} \ldots R^{c_p d_p}_{a_p b_p} \tag{1}$$

where R^{cd}_{ab} is the Riemann tensor and $\delta^{a\ldots b}_{c\ldots d}$ is the generalized Kronecker delta. For $p = 1$, this term is proportional to the Ricci scalar, and for $p = 2$ in $D = 4$, it corresponds to the Gauss-Bonnet term.

A key property of these Lagrangian's is their connection to topology. The generalized Gauss-Bonnet theorem states that for any compact, orientable $2p$-dimensional Riemannian manifold $\mathcal{M}^{2p}$ without boundary, the integral of the p-th Lovelock Lagrangian is a topological invariant proportional to the Euler characteristic, $\chi(\mathcal{M})$:

$$\oint_{\mathcal{M}^{(2p)}} \mathcal{L}_{(2p)}\, d^{2p}x = (4\pi)^p\ p!\ \chi(\mathcal{M}) \tag{2}$$

This expression demonstrates that an integral of a local curvature invariant yields a global topological invariant. In four dimensions ($p = 2$), this reduces to the standard Gauss-Bonnet theorem, where the action is purely topological:

$$S_L = \oint_{\mathcal{M}^4} \mathcal{L}_{(2)} d^4x = 32\pi^2\chi(\mathcal{M}) \tag{3}$$

A separate, powerful result from differential topology, the Poincaré-Hopf theorem, also computes the Euler characteristic. Let $\mathcal{M}$ be a compact, differentiable manifold, and let $\vec{v}_{grav}$ be a smooth vector field on $\mathcal{M}$ manifold with only isolated zeros (points x_i where $\vec{v}_{grav}(x_i) = 0$). The theorem states that the sum of the indices of these zeros is equal to the Euler characteristic of the manifold:

$$\sum_{x_i \in \text{zeros}} \text{Ind}_{x_i}(\vec{v}_{grav}) = \chi(\mathcal{M}) \tag{4}$$

The index $\text{Ind}_{x_i}(\vec{v}_{grav})$ is an integer that quantifies the behavior of the vector field in the immediate neighborhood of the zero x_i. For this theorem to hold, the vector field must satisfy certain regularity conditions:

- Smoothness: The vector field must be of class C^∞, allowing for well-defined derivatives.
- Isolated Zeros: Each zero must be contained within a neighborhood that excludes all other zeros, ensuring the index is well-defined.

By equating these two independent formulations for the Euler characteristic, we establish a direct bridge between spacetime curvature and the structure of vector fields defined upon it:

$$\oint_{\mathcal{M}^{2p}} \mathcal{L}_{(2p)}\, d^{2p}x = (4\pi)^p p! \sum_{x_i \in \text{zeros}} \text{Ind}_{x_i}(\vec{v}_{grav}) \tag{5}$$

This relation provides a powerful tool for analyzing gravitational dynamics. The zeros of the vector field serve as topological probes, revealing global information encoded in the curvature. It is crucial to distinguish these topological singularities (the zeros of $\vec{v}_{grav}$) from physical curvature singularities, where invariants of the Riemann tensor diverge. While distinct, the overall topology of a spacetime, which is determined by its matter-energy content via the field equations, imposes constraints on the types of global vector fields it can admit and, consequently, on the sum of their indices.

This framework allows for the analysis of spacetime topology through the **Poincaré-Hopf index** of a physically motivated vector field, such as a Killing vector field or the gradient of a scalar field [6, 5]. This connection between Lovelock gravity and the Poincaré-Hopf theorem reveals a deep interplay between local geometry and global topology, offering a novel perspective on the structure of spacetime [7, 8].

1.2. Topological Invariants in Gravitational Theories

Topological invariants such as the Euler characteristic $\chi(\mathcal{M})$ play a central role in understanding the structure of spacetime. The Gauss-Bonnet term in four dimensions, $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, integrates to yield a topological invariant: [9]

$$\oint_{\mathcal{M}^4} \mathcal{G}\, \sqrt{-g}\, d^4x = 32\pi^2\chi(\mathcal{M}^4). \tag{6}$$

This is generalized in Lovelock gravity through higher-order Euler densities that remain topological in critical dimensions.

Wei et al. [10] used the Gauss-Bonnet term to classify black hole topologies via topological charge, revealing universal behavior among solutions. Their results suggest a deep ointerplay between the topology of space-time and gravitational dynamics.

1.3. Synthesis and Critical Perspective

Together, these studies suggest a promising direction: combining the formal machinery of Lovelock gravity with topological classification tools like the Poincaré-Hopf index to study singularities, black hole thermodynamics, and cosmological compactification. The use of Euler densities connects the dynamics of the gravitational field with invariant global properties. However, current work often assumes symmetry and idealized configurations (e.g., spherically symmetric or AdS backgrounds), and more research is needed to explore generic, less symmetric cases.

Moreover, the direct connection between the Poincaré-Hopf index and the functional form of Lovelock terms in arbitrary manifolds remains an open field. Developing this connection would not only advance mathematical relativity but may also yield insights into quantum gravity and the role of topology in spacetime emergence.

2. LITERATURE REVIEW

2.1. Lovelock Gravity and Higher-Dimensional Generalizations

Lovelock gravity, formulated by David Lovelock in 1971 [1, 11], provides the most general second-order Euler–Lagrange tensor derivable from a metric in higher dimensions. Its Lagrangian includes higher-order curvature invariants, extending the Einstein-Hilbert action. In four dimensions, the Lovelock terms reduce to the Einstein-Hilbert and Gauss-Bonnet contributions, with the latter being topologically invariant and non-dynamical [3].

Theoretical studies have shown that in even-dimensional spacetime, the Lovelock action contributes dynamically and leads to modified equations of motion [2]. Charmousis [12] emphasizes the physical richness of Lovelock gravity, particularly in black hole thermodynamics and braneworld cosmology. Notably, Lovelock gravity preserves second-order field equations, thus avoiding ghosts and ensuring stability.

3. METHODOLOGIES

3.1. Processing

This research aims to study the connection between Lovelock's theory of gravity, which is considered a generalization of Einstein-Hilbert's action, the Euler index, and the Poincaré-Hopf index, by tracking singularity points in space using smooth directional fields.

These are the steps of the research methodology:

- Study the link between Einstein-Hilbert's action and Lovelock's theory of more general action.
- studying the link between the Euler index and the Poincaré-Hopf index and the relationship between them and the Lovelock action of gravity.
- Exploiting the singularity theory to explain the value of Lovelock's effect on gravity
- Expanding the concept of Einstein-Hilbert's action and its relationship to singularities in space

4. MATHEMATICAL FORMALISM

4.1. Lovelock Gravity

The Lovelock Lagrangian is a sum of dimensionally extended Euler densities [1, 5]. The general form of the Lagrangian in D dimensions is:

$$\mathcal{L} = \sum_{p=0}^{\lfloor (D-1)/2 \rfloor} \alpha_p \mathcal{L}_{(p)} \tag{7}$$

where α_p are coupling constants and $\mathcal{L}_{(p)}$ is the p^{th}-order Lovelock Lagrangian

since :

$$\mathcal{L}_{(p)} = \Omega^{\mu_1\mu_2} \wedge ... \wedge \Omega^{\mu_{2p-1}\ \mu_{2p}} \wedge \epsilon_{\mu_1....\mu_{2p}} \tag{8}$$

$$\Omega^{\mu}_{\eta} = d\Gamma^{\mu}_{\eta} + \Gamma^{\mu}_{\sigma} \wedge \Gamma^{\sigma}_{\eta} \tag{9}$$

and more simply :

$$\Omega^{\mu\eta} = \frac{1}{2} R^{\mu\eta}_{\alpha\beta}\ \omega^{\alpha} \wedge \omega^{\beta} \tag{10}$$

and

$$\epsilon_{\mu_1....\mu_k} = \frac{\epsilon_{\mu_1....\mu_k}}{(d-k)!}\ \omega^{A_{k+1}} \wedge ... \wedge \omega^{A_d} \tag{11}$$

where ω^A is the co-frame dual to the frame to which the indices refer. The integrals of these dimension-ally continued (Euler) forms $\mathcal{L}_p$ called Gauss-Bonnet actions in particular if the metric g_{ij} is taken with Lorentzian signature. In a given dimension there is only a finite number of these terms. The following identity is very helpful:

$$\omega^A \wedge \epsilon_{\mu_1....\mu_k} = \delta^A_{\mu_k} \epsilon_{\mu_1....\mu_{k-1}} - \delta^A_{\mu_{k-1}} \epsilon_{\mu_1...\mu_{k-2}\mu_k} + ... - (-1)^k \delta^A_{\mu_1} \epsilon_{\mu_2....\mu_k} \tag{12}$$

It is sometimes settled in the simplified form:

$$\mathcal{L}_{(p)} = \frac{1}{2^p} \delta^{\mu_1\nu_1...\mu_p\nu_p}_{\rho_1\sigma_1...\rho_p\sigma_p} R^{\rho_1\sigma_1}_{\mu_1\nu_1} \dots R^{\rho_p\sigma_p}_{\mu_p\nu_p} \tag{13}$$

Here, $\delta^{\cdots}_{\cdots}$ is the generalized Kronecker delta tensor.

- For $p = 0$, $\mathcal{L}_{(0)}$ is proportional to the volume form, giving the cosmological constant term.

- For $p = 1$, $\mathcal{L}_{(1)}$ is the Ricci scalar R, giving the standard Einstein-Hilbert action.
- For $p = 2$, $\mathcal{L}_{(2)}$ is the Gauss-Bonnet term: $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$.

A crucial property is that in $D = 2p$ dimensions, the closed integral of $\mathcal{L}_{(p)}$ is a topological invariant proportional to the Euler characteristic $\chi(\mathcal{M}^{2p})$:

$$\oint_{\mathcal{M}^{(2p)}} d^{2p}x\sqrt{-g}\mathcal{L}_{(p)} = (4\pi)^p p!\chi(\mathcal{M}) \tag{14}$$

The basis of Lovelock's theory of gravity begins with a generalization of Gauss-Bonnet's theory of gravity. [11, 13, 8] This is due to the strong similarity between Gauss-Bonnet's theory of curved space and Einstein-Hilbert's action [11, 5, 7], on the other hand,

$$S_{EH} = \oint_{\mathcal{M}^4} \mathcal{L}(g_{\mu\nu}, \Gamma^{\sigma}_{\mu\nu}) \tag{15}$$

$$S_L = \oint_{\mathcal{M}^4} dx^4\sqrt{g}\,\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\eta}R^{\mu\nu\eta}\right) \tag{16}$$

and therefore any theoretical extension is subject to several conditions necessary to build a correct geometric model [6, 14]. On the other hand, **Poincaré-Hopf theory** stands out due to of its connection to Gauss-Bonnet's theory, because of its connection to the topological properties of the Riemannian space.

$$\oint_{\mathcal{M}^4} dx^4\sqrt{g}\,\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\eta}R^{\mu\nu\eta}\right) = 32\pi^2\chi(\mathcal{M}) \tag{17}$$

4.2. Lovelock Equations of motion

we can know the equations of motion by finding the action least of the Lovelock limit in equation Eq. (7), and therefore it is considered more general than those produced by the Einstein-Hilbert action.

$$\delta\mathcal{L}_{2p} = 0 \tag{18}$$

It will be sufficient to treat the equations of motion in four dimensions according to Einstein-Hilbert's action, put $\mathcal{L}_{2p} = \mathcal{L}_2$ in equation Eq. (13)

$$\mathcal{L}_2 = \left(\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\eta}R^{\mu\nu\eta}\right)\sqrt{g} \tag{19}$$

where α, β, γ are arbitrary constants that will be specified later ,and This leads us to

$$\delta\mathcal{L}_2 = \left(\alpha\delta R^2 + \beta\delta(R_{\mu\nu}R^{\mu\nu}) + \gamma\delta(R_{\mu\nu\eta}R^{\mu\nu\eta})\right)\sqrt{g} + \left(\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\eta}R^{\mu\nu\eta}\right)\delta\sqrt{g} \tag{20}$$

we can write the covariance for each of these terms 1^{th} term

$$\delta(R^2\sqrt{g}) = \sqrt{g}\left(2R_{\sigma\eta}R - \frac{1}{2}g_{\sigma\eta}R^2 - 2g_{\sigma\eta}\nabla^2 R + 2\nabla_\eta\nabla_\sigma R\right)\delta g^{\sigma\eta} \tag{21}$$

2^{th} term

$$\delta(R_{\sigma\eta}R^{\sigma\eta}\sqrt{g}) = \sqrt{g}\left(-2R^{\eta}_{\sigma\varphi\chi}R^{\sigma}_{\eta} - \frac{1}{2}g_{\varphi\chi}\nabla^2 R + \nabla_\varphi\nabla_\chi R - \nabla^2 R_{\varphi\chi} - \frac{1}{2}g_{\varphi\chi}R_{\sigma\eta}R^{\sigma\eta}\right)\delta g^{\sigma\eta} \tag{22}$$

3^{th} trem

$$\delta(R_{\sigma\eta\theta\zeta}R^{\sigma\eta\theta\zeta}\sqrt{g}) = \sqrt{g}\left(R_{\sigma\eta\theta\zeta}R^{\sigma\eta\theta\zeta} - 4R_{\sigma\varphi}R^{\sigma}_{\chi} - 4R^{\eta}_{\sigma\varphi\chi}R^{\sigma}_{\eta} + 2R_{\varphi\sigma\eta\theta}R^{\sigma\eta\theta}_{\chi} + 2\nabla_\varphi\nabla_\chi R - 4\nabla^2 R_{\varphi\chi}\right) \tag{23}$$

from the equation Eq. (21) We will put

$$\overset{\circ 1}{G}_{\sigma\eta} = \left(2R_{\sigma\eta}R - \frac{1}{2}g_{\sigma\eta}R^2 - 2g_{\sigma\eta}\nabla^2 R + 2\nabla_\eta\nabla_\sigma R\right) \tag{24}$$

Therefore, we will put the equation Eq. (21) in its new form:

$$\delta(R^2\sqrt{g}) = \sqrt{g}\;\overset{\circ 1}{G}_{\sigma\eta}\;\delta g^{\sigma\eta} \tag{25}$$

and from the equation Eq. (22) We will put

$$\overset{\circ 2}{G}_{\sigma\eta\varphi\chi} = \left(-2R^{\eta}_{\sigma\varphi\chi}R^{\sigma}_{\eta} - \frac{1}{2}g_{\varphi\chi}\nabla^2 R + 2\nabla_\varphi\nabla_\chi R - \nabla^2 R_{\varphi\chi} - \frac{1}{2}g_{\varphi\chi}R_{\sigma\eta}R^{\sigma\eta}\right) \tag{26}$$

and from the equation Eq. (23)

$$\overset{\circ 3}{G}_{\sigma\eta\theta\zeta\varphi\chi} = \left(R_{\sigma\eta\theta\zeta}R^{\sigma\eta\theta\zeta} - 4R_{\sigma\varphi}R^{\sigma}_{\chi} - 4R^{\eta}_{\sigma\varphi\chi}R^{\sigma}_{\eta} + 2R_{\varphi\sigma\eta\theta}R^{\sigma\eta\theta}_{\chi} + 2\nabla_\varphi\nabla_\chi R - 4\nabla^2 R_{\varphi\chi}\right) \tag{27}$$

Therefore, the basic equation of change Eq. (20) will become

$$\alpha\,\overset{\circ 1}{G}_{\varphi\chi} + \beta\overset{\circ 2}{G}_{\sigma\eta\varphi\chi} + \gamma\overset{\circ 3}{G}_{\sigma\eta\theta\zeta\varphi\chi} = 0 \tag{28}$$

After putting these equations into the main formulaEq. (28), the most simplified form of equation

$$\frac{-1}{2\sqrt{g}}g_{\varphi\chi}\mathcal{L}_2 + 2R_{\varphi\chi}R - 4R_{\varphi\sigma}R^{\sigma}_{\chi} + 4R^{\eta}_{\sigma\varphi\chi}R^{\sigma}_{\eta} + 2R_{\varphi\sigma\eta\theta}R^{\sigma\eta\theta}_{\chi} = 0 \tag{29}$$

Therefore, the equations of motion change according to Lovelock's generalization, and from that we can see a connection between the singularities of space related to the Poincaré-Hopf index and the equations of motion in space according to the action of Lovelock's gravity through the formula Eq. (5)

4.3. The Characteristic Curvature Vector Field

The primary challenge in linking Eq. Eq. (14) and Eq. Eq. (4) is to define a physically meaningful vector field $\vec{v}_{grav}$ whose zeros correspond to the topological defects of a given spacetime. The construction of such a field from first principles is a deep problem that we leave for future work.

For the purpose of this paper, we postulate the existence of a **characteristic curvature vector field**, $\vec{v}_{grav}$, with the following properties:

i. It is constructed from the spacetime metric $g_{\mu\nu}$ and its derivatives.
ii. Its isolated zeros, x_i, are located at points of fundamental physical and topological significance (e.g., singularities, event horizons).
iii. The sum of the indices of its zeros equals the Euler characteristic of the manifold, $\sum_{x_i \in \text{zeros}} \mathrm{Ind}_{x_i}(\vec{v}_{grav}) = \chi(\mathcal{M})$, thus allowing it to serve as the vector field in the Poincaré-Hopf theorem for gravitational contexts.

The existence of such a field would establish a direct bridge between the Lovelock action and the local topological structure of spacetime.

5. MAIN RESULT: THE LOVELOCK ACTION AS A TOPOLOGICAL SUM

Although this research paper is not a new scientific breakthrough, it focuses on reinterpreting existing knowledge based on inference from the results of previous research papers. By combining the established relationship between the Lovelock action and the Euler characteristic (Eq. 14) with the Poincaré-Hopf theorem (Eq. 4), and using our postulated characteristic curvature vector field, we arrive at our main result.

For a $2p$-dimensional compact manifold $\mathcal{M}^{2p}$, the integrated p^{th}-order Lovelock action can be expressed as a sum over the indices of the zeros of the characteristic curvature vector field:

$$\oint_{\mathcal{M}^{2p}} d^{2p}x\sqrt{g}\mathcal{L}^{(2p)} = (4\pi)^p p! \sum_{x_i \in \text{zeros}} \mathrm{Ind}_{x_i}(\vec{v}_{grav}) \quad (30)$$

This equation is the central thesis of our paper. It suggests that the value of the gravitational action, a quantity central to dynamics via the principle of least action, is fundamentally determined by the number and nature of the topological defects within the spacetime. This implies a profound conservation principle: since the right-hand side is a sum of integers fixed by the topology, the value of the action is a topological invariant and does not change under smooth deformations of the geometry that do not create or destroy topological defects.

6. APPLICATION

6.1. The Schwarzschild Black Hole

To demonstrate the plausibility and implications of our framework, we apply it to the 4-dimensional spacetime of a Schwarzschild black hole. For $D = 4$, the relevant Lovelock term is the Gauss-Bonnet term $\mathcal{L}^{(2)}$. Its integral over the manifold is:

$$\oint_{\mathcal{M}^4} d^4x\sqrt{-g}\mathcal{L}_{(2)} = 32\pi^2\chi(\mathcal{M}) \quad (31)$$

To evaluate this, we need the Euler characteristic of the Schwarzschild spacetime. When properly regularized to handle the singularity and the asymptotic region, the Euler characteristic of the exterior region of a non-extremal black hole is known to be $\chi(\mathcal{M}) = 2$ [10]. This integer can be thought of as corresponding to the single topological defect of the black hole itself.

Now, we invoke our postulate. We assume the existence of a characteristic vector field $\vec{v}_{grav}$ for the Schwarzschild spacetime. According to our framework, this field should have zeros whose indices sum to $\chi(\mathcal{M}) = 2$. A natural assumption is that the black hole itself corresponds to a single topological defect with a zero located, for instance, at the event horizon. For the sum of indices to be 2, this zero must have an index of $+2$. This is analogous to the index of a vector field on the surface of a sphere.

With this assumption, we can apply our main result (Eq. 30) for $p = 2$:

$$\begin{aligned}\oint_{\mathcal{M}^4} d^4x\sqrt{-g}\mathcal{L}^{(2)} &= (4\pi)^2 2! \sum_{x_i \in \text{zeros}} \mathrm{Ind}_{x_i}(\vec{v}_{grav}) \\ &= 32\pi^2 \times (+2) \\ &= 64\pi^2\end{aligned} \quad (32)$$

This result, derived from the topological index, is consistent with the direct calculation using the Euler characteristic. It demonstrates that if a vector field with the required topological properties exists, our framework provides a consistent method for calculating the value of the topological part of the gravitational action.

7. FUTURE APPLICATIONS AND RESEARCH DIRECTIONS

The established intrinsic connection between the Lovelock gravitational action and the Euler characteristic, formally expressed by the Poincaré-Hopf index sum

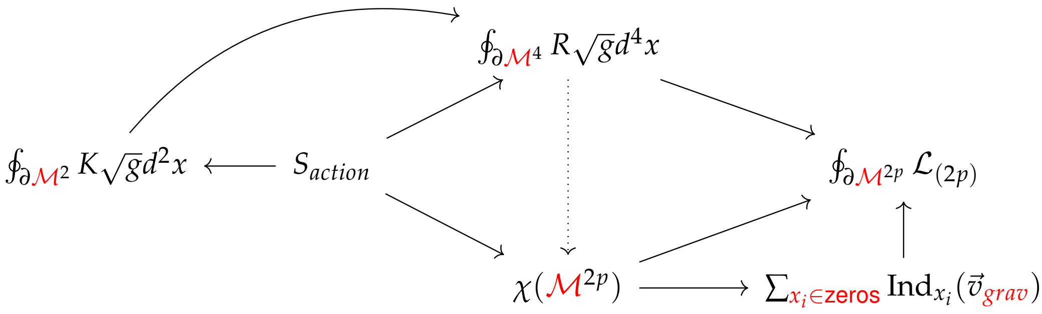

Figure 1. An indirect relationship (the dashed line) connects the action of Einstein-Hilbert and the Euler index, starting with the Gauss-Bonnet action and moving on to the more generic action of Hilbert-Einstein and then to the more general action of Lovelock. As a result, we may use the Poincaré-Hopf theory and the generalized Stokes theorem to go from this link to Lovelock's more generic act. The Poincaré-Hopf index connects the Lovelock action with the Euler index.

$\sum Ind_{x_i}(\vec{v}_{\text{grav}})$, provides a novel topological framework for studying spacetimes. Building upon the central relation (Eq. (30) in the main text), which links the action to the topological invariant, the following applications are proposed as future research avenues to deepen our understanding of gravitational dynamics and the properties of spacetime singularities.

7.1. Topological Classification of Generalized Black Hole Solutions

This application aims to transcend the conventional classification of black holes—based on physical parameters such as mass (M) and charge (Q)—by substituting it with a framework rooted in the inherent topological properties of the spacetime manifold, particularly within the context of generalized Lovelock gravity.

- **Methodology:** We propose the study of black hole solutions in higher-order Lovelock gravity ($p > 2$) in dimensions $D \geq 6$. The core idea is to utilize the index sum $\sum Ind_{x_i}(\vec{v}_{\text{grav}})$, i.e., the Euler characteristic $\chi(\mathcal{M})$, as a **Topological Charge** intrinsic to the gravitational solution.
- **Connecting Topology to Physics:** A crucial step involves explicitly identifying the zeros x_i of the characteristic vector field $\vec{v}_{\text{grav}}$ and demonstrating their correspondence to physical topological defects, such as the event horizon and the central singularity. The local index Ind_{x_i} at the singularity could potentially classify the nature of the singularity (e.g., a source or a saddle point).
- **Expected Outcome:** This approach promises to yield a robust, integer-valued classification system for organizing and interpreting the diverse spectrum of black hole solutions arising from generalized gravitational theories.

7.2. Stability Analysis of Gravitational Solutions via the Poincaré-Hopf Index

This research direction seeks to establish a correlation between the local topological properties (the Poincaré-Hopf index) and the global dynamical stability of Lovelock gravity solutions.

- **Study of Extremal Black Holes:** Extremal black holes, which represent critical points or phase transitions in black hole thermodynamics, are ideal candidates for this analysis. We hypothesize that the transition to the extremal state (e.g., as temperature approaches zero) corresponds to a **change in the local index** of a zero of the field $\vec{v}_{\text{grav}}$ located at the horizon.
- **Index as a Stability Measure:** In dynamical systems theory, fixed points with negative indices (saddle points) are typically associated with instability. By extension, establishing a physical definition for $\vec{v}_{\text{grav}}$ could allow the sign of Ind_{x_i} to serve as a **topological stability criterion** for the gravitational solution (stable/unstable).
- **Expected Outcome:** The development of a topological criterion for stability would offer a profound and geometrically general method for examining the physical viability of complex gravitational solutions, complementing traditional perturbation analyses.

7.3. The Role of Topological Action in Quantum Gravity Formalisms

This application explores the implications of the topological invariance of the Lovelock action, especially in its critical dimension $D = 2p$, within the framework of quantum gravity.

- **Action Quantization:** The relation $\oint_{\mathcal{M}} \mathcal{L}^{(p)} \propto \chi(\mathcal{M})$ implies that the action in $D = 2p$ is inherently quantized, taking values proportional to the integer Euler characteristic. This suggests that the value of this component of the action is invariant under smooth metric fluctuations.

- **Regularization in Quantum Field Theory (QFT):** The topological nature of the term suggests that Lovelock terms (e.g., the Gauss-Bonnet term in $D = 4$) could play a fundamental role in **regularizing** the UV divergences of quantum gravity. Since their contribution to the path integral is independent of local metric variations, they could stabilize the theory at high energies without introducing ghost modes.
- **Expected Outcome:** This viewpoint reinforces the idea that topological invariants are more fundamental than local metric details at high energies, guiding research toward a topologically constrained formulation of quantum spacetime.

8. DISCUSSION AND CONCLUSION

We have proposed a novel framework connecting Lovelock gravity to the topology of spacetime through the Poincaré-Hopf theorem. The central idea is that the Lovelock action can be expressed as a sum of topological indices associated with a characteristic vector field on the manifold. This reframes a key element of gravitational dynamics as a consequence of the underlying spacetime topology.

The primary strength of this proposal is its explanatory power. It suggests that the value of the gravitational action is determined by a discrete set of topological defects, providing a new perspective on conservation laws and the quantization of spacetime properties. We have shown that this framework is consistent when applied to the well-understood case of a Schwarzschild black hole, assuming a vector field with the appropriate topological properties exists.

The main limitation, and the most crucial direction for future work, is the construction of the characteristic curvature vector field, $\vec{v}_{grav}$, from first principles. Such a construction would need to be physically motivated and mathematically rigorous, likely involving a complex interplay of curvature invariants. Solving this problem would be a major step towards a full theory realizing the ideas presented here.

1. Detecting Singularities Using the Poincaré-Hopf Index

If we have a more complex surface, such as a surface with real singularities (such as points where the amount of curvature is infinite), we can use a pointer to select these singularities. Near points where the curvatures are extreme (real singularities), the Poincaré-Hopf index becomes non-zero and can be used to locate the singularities.

This result does not mean the absence of uniqueness points, but rather it means that the sum of uniqueness points cancel each other out. This is illustrated by the definition of the Poincaré-Hopf index in Eq. (4), during the given manifold $\mathcal{M}^4$.

$$\chi(\mathcal{M}) = \sum_{x_i \in \text{zeros}} \text{Ind}_{x_i}(\vec{v}_{grav}) = 0 \tag{33}$$

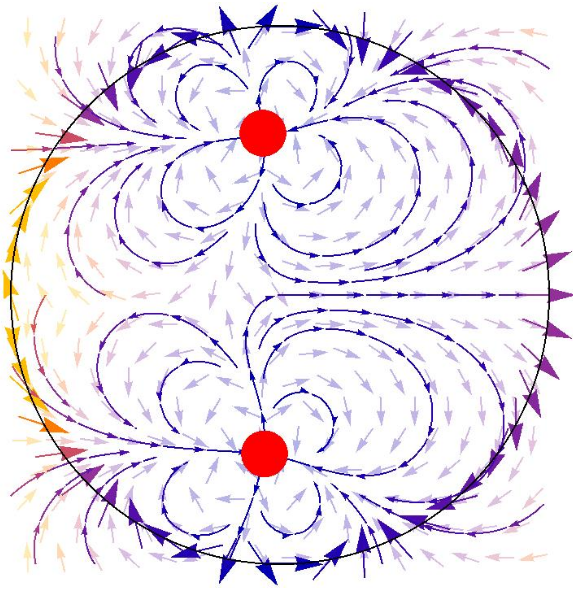

Figure 2. The net vector field around any closed path with isolated poles depends on what kind of poles they are, either sources or drains. The vector field rotates around the closed path several times until it returns to its original state. This means that the net curvature of space is disappearing and is equal to zero. So, it also decides what kind of isolated poles are in the closed path. The Poincaré-Hopf index theorem states that their sum is $\sum_{x_i \in \text{zeros}} \text{Ind}_{x_i}(\vec{v}_{grav}) = 0$

2. Relationship of Poincaré-Hopf theory with Lovelock's gravity:

Using the idea of the relationship between the Einstein-Hilbert action and the Chern-Gauss-Bonnet curvature, the Poincaré-Hopf index was connected to Lovelock's theory of gravity. Because of this direct relationship, the singularity sites in space limit the value of the action without delving into the specifics of integration in order to identify the action. This is because they reflect a deeper understanding of the Einstein-Hilbert action (Lovelock's gravity). This not only clearly shows that the value of the integral for Lovelock's gravity is constant, but it also clearly shows that either Lovelock's gravity or Einstein's action is a constant quantity, which inevitably reflects the conservation and symmetry principles.

$$\oint_{\mathcal{M}^4} dx^4 \sqrt{g}\, \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\eta}R^{\mu\nu\eta}\right) = 32\pi^2 \sum_{x_i \in \text{zeros}} \text{Ind}_{x_i}(\vec{v}_{grav}) \tag{34}$$

Therefore, Equation Eq. (17) indicates that the change in Lovelock's action with the passage of time will be zero due to the right side of Equation Eq. (34), which does not seem to depend on the passage of time.

$$\frac{d\mathcal{L}_2}{dt} = 0 \tag{35}$$

On the other hand, figure 2 shows that the integral of Lovelock's gravitational action is close to zero, and this property is topologically equivalent to the surface of a donut, the surface of an infinite cylinder, or the surface of a Euclidean plane, as the common factor between Euclidean and Riemannian surfaces is the Poincaré-Hopf index for the field spread over a certain surface, and this means The smooth directional field on Euclidean surfaces leads to the lack of value of the indicator due to the absence of singularity points on that surface.

3. Singularities and Poincaré-Hopf theory :

The concept of singularities in space is a sensitive and important field because it characterizes space and reflects its essential properties and thus directly affects the equations of motion for bodies. Discovering singularities is a very important matter using various methods. Therefore, an attempt was made to link finding singularities using the Poincaré-Hopf index.

It is clear from Equation Eq. (17) that we can express the value of the integration in terms of the resultant of the sources and saddle points within the points of the closed manifold $\mathcal{M}^4$.

$$\oint_{\mathcal{M}^4} dx^4 \sqrt{g} \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\eta}R^{\mu\nu\eta} \right) = 32\pi^2 \sum_{x_i \in \text{zeros}} \text{Ind}_{x_i}(\vec{v}_{grav}) \tag{36}$$

Therefore, this result indicates, in one way or another, that Lovelock's theory of gravity, which in its most general state is considered to be Einstein-Hilbert's action, depends directly on singularities confined within a closed path, and therefore, on the other hand, it reflects the invariance of the action, and this indicates the existence of a symmetry contained in Lovelock's action of gravity.

In conclusion, this research provides a new and promising path for exploring the deep relationship between gravity, geometry, and topology.

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