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Neutrosophic Cosets and Neutrosophic Normal Subgroups of Neutrosophic Groups Associated With $G_1^t\,[I]$

Adel Mohammed Al-Odhari

Faculty of Education, Humanities and Applied Sciences (Khawlan) and Department of Foundations of Sciences, Faculty of Engineering, Sana'a University. Box:13509, Sana'a, Yemen.

*Corresponding author: a.aleidhri@su.edu.ye

ABSTRACT

The Article discusses neutrosophic left/right cosets, their properties, neutrosophic normal subgroups, and neutrosophic quotient groups with some theories and examples. The concept of Neutrosophic Groups was introduced by Kandasamy and Smarandache in their work in 2006 as part of a broader field of research in neutrosophy. Neutrosophic Groups are defined by classical NeutroAxioms according to the Neutrosophic Set theory, which has type-1, The paper explores various structures related to Neutrosophic Groups, including examples of neutrosophic groups, neutrosophic left/right cosets, neutrosophic normal subgroups, Neutrosophic Lagrange's theorem, quotient groups, and their properties, several results of the theorems, and examples are demonstrated with $G_1^t\,[I]$.

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1. INTRODUCTION

Neutrosophic science is a modern branch of mathematics that was born in the last three decades, approximately by Smarandache. He developed the concept of degree membership function from the intuitionistic fuzzy set into a neutrosophic set. For a summary of the development of the neutrosophic and plithogenic literature theories, we recommend referring to a recent paper entitled " Three Decades of Neutrosophic and Plithogenic Theories with their Applications (1995 - 2024)." in [1]. The neutrosophic group theory appeared for the first time in 2006 by Kandasamy and Smarandache as an extension of classical group theory in [2], and [3] respectively. At different and successive periods, another group of researchers joined to study the neutrosophic groups as follows: in 2012, Agboola, Akwu, and Oyebo in their article Neutrosophic groups and subgroups in [4]. In 2019, Abobala, Hatip, and Alhamido published a contribution to neutrosophic groups in [5]. In 2020, Agboola presented the article Introduction to Neutrosophic Groups in [6]. In 2021, Rozina published a review study on neutrosophic groups and their generalizations in [7]. In 2024, Al-Odhari wrote the axiomatic neutrosophic groups and neutrosophic subgroups for the axiomatic neutrosophic groups in [8], and [9] and another humble contribution to neutrosophic linear algebra in [10], and [11]. Moreover, the author asks himself if there is a theory of neutrosophic sets similar to the theory of traditional or classical set theory. To answer this question, we established a new foundation of the neutrosophic set theory of three types in [12–17]. This work is being carried out in parallel with work on the theory of nitrosophic rings in [18–20]. This article presents neutrosophic cosets, neutrosophic normal subgroups, and neutrosophic quotient groups with their properties.

2. Neutrosophic Groups and Neutrosophic Subgroups According to Neutrosophic Set Theory of $G_1^t\,[I]$

In this section, we presented the neutrosophic binary operations on neutrosophic sets of three types with their properties, and I will rethink neutrosophic groups according to the classification of neutrosophic sets of three types, and using the symbolization of them, such as our work in

[12], [17], and [13]. The concept of a neutrosophic binary operation is very important in neutrosophic abstract algebra and other structures of neutrosophic mathematics. So, I will define the neutrosophic binary operations associated with a neutrosophic set of three types related to neutrosophic groups and examine their basic properties. Moreover, I introduce some examples of neutrosophic groups and neutrosophic subgroups.

Definition 2.1[12] Let $G \neq \emptyset \subset U$ be a nonempty classical set, then:

1. $G_1^t[I] = \{g_1 + g_2 I : h_1, h_2 \in G\}$ is a neutrosophic set of type-1,
2. $G_2^t[I] = \{gI \cup \{g\} : g \in G\}$ is a neutrosophic set of type-2,
3. $G_3^t[I] = \{(g_1 + g_2 I) \cup \{g_1\} : g_1, g_2 \in G\}$ is a neutrosophic set of type-3, where I is an indeterminacy.

The perception of the neutrosophic set of type one $G_1^t[I]$. It goes back to Kandasamy and Smarandache in [2, 3], while the neutrosophic set of type three, we propose to treat it with the classical set G, when G does not contain zero. However, we have proposed a perception of a neutrosophic set of the second type, some of the concepts related to which were studied in [12–17]. It may also allow for the construction of some neutrosophic algebraic properties. Of course, one can propose the neutrosophic set using the concept of indeterminacy I, where $I^2 = I$, $0.I = 0$, and $1.I = 1$.

Definition 2.2 [2]Let $(G, *)$ be any group, and $\langle G \cup I \rangle$ is given by:

$$\langle G \cup I \rangle = \{a + bI : a, b \in G\}.$$

Then the neutrosophic algebra structure $N(G) = \{\langle G \cup I \rangle, *\}$ is called the neutrosophic group, which is generated by I and G under $*$.

Theorem 2.1[2]Let$(G, *)$ be a group, $N(G) = \{\langle G \cup I \rangle, *\}$ be the neutrosophic group, then:$N(G)$. In general, it is not a group, and $N(G)$ always contains a group.

In the remainder of this section, we will review and examine previous work on the neutrosophic group in [8, 9] and its relationship to the work of the nitrosophic sets of three types in [12–17]. To improve this paper in a way that serves the community of neutrosophic knowledge.

Definition 2.3 [8] Let $G \neq \emptyset \subset U$ be a nonempty classical set, and $G_1^t[I] = \{a + bI : a, b \in G\}$ is a neutrosophic set of type-1. Consider $*$ is a Neutrosophic binary operation on $G_1^t[I]$. Then the nautrosophic pair $\langle G_1^t[I], * \rangle$ is called the neutrosophic group, which is generated by I if it satisfies the axiomatic conditions of a group:
NG$_1$: For all x, y and $z \in N(G)$, $(x * y) * z = x * (y * z)$ " associative law";
NG$_2$: There exists $e_N = e + eI \in N(G)$ such that for all $x * e_N = x = e_N * x$" existence of an identity" and;
NG$_3$: For all $x \in N(G)$, there exists $y \in N(G)$ such that $x * y = e_N = y * x$" existence of inverse ". Thus, a neutrosophic group is a neutrosophic mathematical system $N(G) = \langle G_1^t[I], * \rangle$ satisfying the axioms NG$_1$ to NG$_3$. Otherwise, it is called a neutrosophic algebra structure.

Observations. Here, we modified Definition 1.3 in[8]. This means that one can be perceptive and try to define a binary operation on $G_1^t[I]$ and check Defintion 1.3. Let us claim that Definition 1.3 still works when replacing $G_1^t[I]$ by $G_2^t[I]$ or $G_3^t[I]$. We recall that $G_1^t[I] = G_3^t[I]$ iff G contains zero [12].

Theorem 2.2 If $(G, *)$ is a classical group, then $(G_1^t[I], *)$ is a neutrosophic group generated by I from G as a neutrosophic set of type-1 under the same operation $*$.

Proof. Assume that $x, y, z \in G_1^t[I]$. To check Definition 1.3.

1. Consider $x, y \in G_1^t[I] \Longrightarrow \exists x_1, x_2, y_1, y_2 \in G$ and indeterminacy I such that $x = x_1 + x_2 I$ and $y = y_1 + y_2 I$
$\Longrightarrow \exists x_1\ y_1, x_2 y_2 \in G$ and indeterminacy I such that $xy = x_1 y_1 + x_2 y_2 I$
$\Longrightarrow xy \in G_1^t[I]$, so, $*$ is a closure operation on $G_1^t[I]$.
2. Consider $x, y, z \in G_1^t[I] \Longrightarrow \exists (x_1 y_1), z_1, (x_2 y_2), z_2 \in G$ and indeterminacy I such that $(xy) = (x_1 y_1) + (x_2 y_2) I$, and $z = z_1 + z_2 I$ $\Longrightarrow \exists (x_1 y_1) z_1, (x_2 y_2) z_2 \in G$ and indeterminacy I such that
$$(xy) z = (x_1 y_1) z_1 + (x_2 y_2) z_2 I$$
$$\Longrightarrow (xy) z = (x_1 y_1) z_1 + (x_2 y_2) z_2 I \in G_1^t[I]$$
$\Longrightarrow (xy) z = x (yz) = \underbrace{x_1 (y_1 z_1)}_{\in G} + \underbrace{x_2 (y_2 z_2)}_{\in G} I \in G_1^t[I]$. Hence, $*$ is an associative operation on $G_1^t[I]$.
3. $\exists e_N \in G_1^t[I]$ such that for all $x \in G_1^t[I] \Longrightarrow \exists x_1, x_2, e_1, e_2 \in G$, and indeterminacy I such that
$$e_N * x = (e_1 + e_2 I) * (x_1 + x_2 I)$$
$$= (e_1 * x_1) + (e_2 * x_2) I$$
$= x_1 + x_2 I = x$, and
$$x * e_N = (x_1 + x_2 I) * (e_1 + e_2 I)$$
$$= (x_1 * e_1) + (x_{2*} e_2) I$$
$= x_1 + x_2 I = x$. Since the identity is unique in G, we have $e_1 = e_2$.
4. $\forall\ x \in G_1^t[I] \Longrightarrow \exists x_1, x_2 \in G$ and indeterminacy I such that $x = x_1 + x_2 I \Longrightarrow \exists {x_1}^{-1}, {x_2}^{-1} \in G$ and

indeterminacy I such that

$$x^{-1} = {x_1}^{-1} + {x_2}^{-1} I \in G_1^t\,[I]$$

$\Longrightarrow \ \exists x_1 {x_1}^{-1} = e_1, x_2 {x_2}^{-1} = e_2 \in G$ and indeterminacy I such that

$$x x^{-1} = x_1 {x_1}^{-1} + x_2 {x_2}^{-1} I = e_1 + e_2 I = e_N \in G_1^t\,[I].$$

Corollary 2.3 If $(G, *)$ is an abelian group, then $\left(G_1^t\,[I\,], *\right)$ is a neutrosophic abelian group generated by I and G as a neutrosophic set of type-1 under the same operation $*$.

Proof. It follows directly from Theorem 1.2.

Theorem 2.4 Let H be a subgroup of a group G, then $H_1^t\,[I]$ is a neutrosophic subgroup of $G_1^t\,[I]$ under the same operation, where $H_1^t\,[I]$ is generated by I by H, and $G_1^t\,[I]$ is generated by G.

Proof. Suppose that $H \preccurlyeq G$, to show that $H_1^t\,[I] \underbrace{\preccurlyeq}_{N} G_1^t\,[I]$.

Let $x, y \in H_i^t\,[I] \Longrightarrow \exists x_1, x_2, y_1, y_2 \in H$ and indeterminacy I such that

$$x = x_1 + x_2 I \text{ and } y = y_1 + y_2 I$$

$\Longrightarrow \exists x_1, x_2, {y_1}^{-1}, {y_2}^{-1} \in H$ and indeterminacy I such that

$$x = x_1 + x_2 I \text{ and } y^{-1} = {y_1}^{-1} + {y_2}^{-1} I$$

$\Longrightarrow \exists x_1\ {y_1}^{-1}, x_2 {y_2}^{-1} \in H$ and indeterminacy I such that $x\ y^{-1} = x_1 {y_1}^{-1} + x_2 {y_2}^{-1} I$

$\Longrightarrow x\ y^{-1} \in H_i^t\,[I] \Longrightarrow H_1^t\,[I] \underbrace{\preccurlyeq}_{N} G_1^t\,[I]$. By Theorem 3.3 in [9].

Example 2.1 Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ be the set of integer numbers. Then the neutrosophic integers of type-1 is given by:

$$\mathbb{Z}_1^t[I] = \left\{ \begin{array}{cccc} 0, & 0 \pm I, & 0 \pm 2I, & 0 \pm 3I, \ \cdots \\ \pm 1, & \pm 1 \pm I, & \pm 1 \pm 2I, & \pm 1 \pm 3I, \ \cdots \\ \pm\ 2, & \pm 2 \pm I & \pm 2 \pm 2I & \pm 2 \pm 3I, \ \cdots \\ \vdots & \vdots & \vdots & \vdots \ \ \ldots \end{array} \right\}.$$

Since $(\mathbb{Z}, +)$ is an abelian group under the usual addition on $\mathbb{Z}$, then $\left(\mathbb{Z}_1^t\,[I], +\right)$ is a neutrosophic abelian group under the usual addition on $\mathbb{Z}_1^t\,[I]$ of the neutrosophic set of type-1. Consider the set $H \subset \mathbb{Z}$, where $H = 3\mathbb{Z} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$. Then the neutrosophic set $H_1^t\,[I\,]$ of type-1 is given by:

$$H_1^t\,[I\,] = \{x + yI : x, y \in H\}$$

$= \{0, \pm 3I, \pm 6I, \ldots, 3, 3 \pm 3I, 3 \pm 6I, \ldots, 6, 6 \pm 3I, 6 \pm 6I, \ldots, n, n \pm 3I, n \pm 6I, \ldots\}$. It is clear that $H_1^t\,[I\,] \underbrace{\preccurlyeq}_{N} \mathbb{Z}_1^t[I]$.

Suppose that we take the neutrosophic integer set of type-2.

$$\mathbb{Z}_2^t[I] = \left\{ \begin{array}{cc} 0, & 0I, \\ \pm 1, & \pm 1I, \\ \pm 2, & \pm 2I \\ \vdots & \vdots \end{array} \right\}$$

Under the usual neutrosophic addition. It does not form a neutrosophic binary operation because if 2 and $3I$ are two elements in $\mathbb{Z}_2^t[I]$, then $2 + 3I \notin \mathbb{Z}_2^t[I]$. **Example 2.2** Let $G = \{a, b\}$ be a classical set with the binary operation given by Table (1).

Table 1. of the binary operation

$*$	a	b
a	a	b
b	b	a

Form a group. Consider $G_1^t\,[I\,]$ is a neutrosophic set of type-1, then

$$G_1^t\,[I\,] = \{g_1 + g_2 I : g_1, g_2 \in G\} = \left\{ \begin{array}{cc} a + aI, & a + bI, \\ b + aI, & b + bI \end{array} \right\}.$$

Define the neutrosophic binary operation $*$ on $G_1^t\,[I\,]$ by the following table:

Table 2. of the neutrosophic binary operation

$*$	$a + aI$	$a + bI$	$b + aI$	$b + bI$
$a + aI$	$a + aI$	$a + bI$	$b + aI$	$b + bI$
$a + bI$	$a + bI$	$a + aI$	$b + bI$	$b + aI$
$b + aI$	$b + aI$	$b + bI$	$a + aI$	$a + bI$
$b + bI$	$b + bI$	$b + aI$	$a + bI$	$a + aI$

It is clear that from Table(2). $*$ is a neutrosophic binary operation, and associative on $G_1^t\,[I\,]$. In addition, the neutrosophic identity $e_N = e + eI = a + aI$, and the neutrosophic inverse element are shown in Table(3).

Table 3. of the inverse neutrosophic elements

$(g_1 + g_2 I)$	$a + aI$	$a + bI$	$b + aI$	$b + bI$
$(g_1 + g_2 I)^{-1}$	$a + aI$	$a + bI$	$b + aI$	$b + bI$

According to the previous argument, the neutrosophic order of Paris $N(G) = \left(G_1^t\,[I\,], *\right)$ forms a commutative neutrosophic group. Let $H = \{a\}$ be a classical subgroup of G. Consider $H_1^t\,[I\,]$ is a neutrosophic set of type-1, then $H_1^t\,[I\,] = \{h_1 + h_2 I : h_1, h_2 \in H\} = \{a + aI\}$. Define the neutrosophic binary operation $*$ on $H_1^t\,[I\,]$ by the following table:

Table 4. of the neutrosophic binary operation

$*$	$a + aI$
$a + aI$	$a + aI$

It is obvious that $H_1^t\,[I\,]$ is a neutrosophic subgroup of $G_1^t\,[I\,] = \left\{ \begin{array}{cc} a + aI, & a + bI, \\ b + aI, & b + bI \end{array} \right\}$ by Theorem 3.5 in [9].

Example 2.3 Let $G = \{1, -1, i, -i\}$ be a classical set of complex numbers under a multiplication of complex

Table 5. for the closure of the neutrosophic binary operation.

*	1+I	1-I	1+iI	1-iI	-1+I	-1-I	-1+iI	-1-iI	i+I	i-I	i+iI	i-iI	-i+I	-i-I	-i+iI	-i-iI
1+I	1+I	1-I	1+iI	1-iI	-1+I	-1-I	-1+iI	-1-iI	i+I	i-I	i+iI	i+iI	-i+I	-i-I	-i+iI	-i-iI
1-I	1-I	1+I	1-iI	1+iI	-1-I	-1+I	-1-iI	-1+iI	i-I	i+I	i+iI	i+iI	-i-I	-i+I	-i-iI	-i+iI
1+iI	1+iI	1-iI	1-I	1+I	-1+iI	-1-iI	-1+I	-1-I	i+iI	i+iI	i-I	i+I	-i+iI	-i-iI	-i-I	-i+I
1-iI	1-iI	1+iI	1+I	1-I	-1-iI	-1+iI	-1+I	-1-I	i+iI	i+iI	i+I	i-I	-i-iI	-i+iI	-i+I	-i-I
-1+I	-1+I	-1-I	-1+iI	-1-iI	1+I	1-I	1+iI	1-iI	-i+I	-i-I	-i+iI	-i-iI	i+I	i-I	i+iI	i+iI
-1-I	-1-I	-1+I	-1-iI	-1+iI	1-I	1+I	1-iI	1+iI	-i-I	-i+I	-i-iI	-i+iI	i-I	i+I	i+iI	i+iI
-1+iI	-1+iI	-1-iI	-1-I	-1+I	1+iI	1-iI	1-I	1+I	-i+iI	-i-iI	-i-I	-i+I	i+iI	i+iI	i-I	i+I
-1-iI	-1-iI	-1+iI	-1+I	-1-I	1-iI	1+iI	1+1	1-I	-i-iI	-i+iI	-i+I	-i-I	i+iI	i+iI	i+I	i-I
i+I	i+I	i-I	i+iI	i+iI	-i+I	-i-I	-i+iI	-i-iI	-1+I	-1-I	-1+iI	-1-iI	1+I	1-I	1+iI	1-iI
i-I	i-I	i+I	i+iI	i+iI	-i-I	-i+I	-i-iI	-i+iI	-1-I	-1+I	-1-iI	-1+iI	1-I	1+I	1-iI	1+iI
i+iI	i+iI	i+iI	i-I	i+I	-i+iI	-i-iI	-i-I	-i+I	-1+iI	-1-iI	-1-I	-1+I	1+iI	1-iI	1-I	1+I
i-iI	i+iI	i+iI	i+I	i-I	-i-iI	-i+iI	-i+I	-i-I	-1-iI	-1+iI	-1+I	-1-I	1-iI	1+iI	1+1	1-I
-i+I	-i+I	-i-I	-i+iI	-i-iI	i+I	i-I	i+iI	i+iI	1+I	1-I	1+iI	1-iI	-1+I	-1-I	-1+iI	-1-iI
-i-I	-i-I	-i+I	-i-iI	-i+iI	i-I	i+I	i+iI	i+iI	1-I	1+I	1-iI	1+iI	-1-I	-1+I	-1-iI	-1+iI
-i+iI	-i+iI	-i-iI	-i-I	-i+I	i+iI	i+iI	i-I	i+I	1+iI	1-iI	1-I	1+I	-1+iI	-1-iI	-1-I	-1+I
-i-iI	-i-iI	-i+iI	-i+I	-i-I	i+iI	i+iI	i+I	i-I	1-iI	1+iI	1+1	1-I	-1-iI	-1+iI	-1+I	-1-I

numbers form a group. Consider $G_1^t\,[I\,]$ is a neutrosophic set of type-1, then:

$$G_1^t\,[I\,]=\{g_1{+}g_2I{:}g_1,g_2{\in}G\}.$$

$$=\left\{\begin{array}{cccc} 1+1I, & 1-1I, & 1+iI, & 1-iI, \\ -1+1I, & -1-1I, & -1+iI, & -1-iI \\ i+1I & i-1I & i+iI & i-iI \\ -i+1I & -i-1I & -i+iI & -i-iI \end{array}\right\}.$$

Let $(g_1{+}g_2I)\,,(g'_1{+}g'_2I) \in G_1^t\,[I\,]$ such that $(g_1+g_2I)*((g'_1+g'_2I)) = ((g_1\bullet g'_1)+(g_2\bullet g'_2)\,I)$, where $\bullet$ is a neutrosophic multiplication of complex numbers.The binary operation as shown in Table(5).
According to Table(5), $*$ is closed, since $g_1\bullet g'_1$ and $g_2\bullet g'_2$ are closed in G. $*$ is associative under $\bullet$.
The neutrosophic element $e_N=1+I$. Each element has a neutrosophic inverse element, as shown in Table(6).

Table 6. for the neutrosophic elements.

g	1+I	1-I	1+iI	1-iI	-1+I	-1-I	-1+iI	-1-iI	i+I	i-I	i+iI	i-iI	-i+I	-i-I	-i+iI	-i-iI
g^{-1}	1+I	1-I	1-iI	1+iI	-1+I	-1-I	-1+iI	-1+iI	-i+I	-i-I	-i-iI	-i+iI	i+I	i-I	i-iI	i+iI

It is clear that the neutrosophic order of pari $N(G) = \left(G_1^t\,[I\,]\,,*\right)$ forms commutative neutrosophic groups. Let $H=\{1,-1\}$ be a classical set, consider $H_1^t\,[I\,]$ is a neutrosophic set of type-1, then

$$H_1^t\,[I\,]=\{h_1{+}h_2I{:}h_1,h_2{\in}H\}.$$

$$=\left\{\begin{array}{cc} 1+I, & 1-I, \\ -1+I, & -1-I \end{array}\right\}.$$

Let $(h_1{+}h_2I)\,,\,(h'_1{+}h'_2I)\in H_1^t\,[I\,]$ such that

$$(h_1{+}h_2I)\,,*\left(h'_1{+}h'_2I\right)=\left(\left(h_1h'_1\right)+\left(h_2h'_2\right)I\right).$$

The table is shown in Table(7).

Table 7. of the closure neutrosophic binary operation

$\bullet$	$1+I$	$1-I$	$-1+I$	$-1-I$
$1+I$	$1+I$	$1-I$	$-1+I$	$-1-I$
$1-I$	$1-I$	$1+I$	$-1-I$	$-1+I$
$-1+I$	$-1+I$	$-1-I$	$1+I$	$1-I$
$-1-I$	$-1-I$	$-1+I$	$-1-I$	$1+I$

Hence, by Theorem 3.5 in [9]. We deduced that $H_1^t\,[I]$ is a neutrosophic subgroup of $G_1^t\,[I]$. Note that the neutrosophic order $\psi\left(H_1^t\,[I]\right)=4,\psi\left(G_1^t\,[I]\right)=16$, and $\psi\left(H_1^t\,[I]\right)$ is divide $\psi\left(G_1^t\,[I]\right)=16$. Define $*{:}G_1^t\,[I]\times G_1^t\,[I]\longmapsto G_1^t\,[I]$ such that

$$\begin{aligned} x*y &= (x_1+x_2I)*(y_1+y_2I) \\ &= (x_1\ y_1)+(x_1\ y_2+x_2y_1+x_2y_2)\,I, \end{aligned}$$

$\forall\ x,y\in G_1^t\,[I\,]$, where $\bullet$ is a multiplication of complex numbers. $*$ is not a neutrosophic binary operation on $G_1^t\,[I\,]$, because, $(1+1I)*(1+1I)=1+3I\notin G_1^t\,[I\,]$.

Example 2.4 Let $G=\{1,-1,i,-i\}$ be a classical set of complex numbers. Consider $G_2^t\,[I]$ is a neutrosophic set of type-2, then

$$G_2^t[I] = \{gI \cup \{g\} : g \in G\}$$

$$= \left\{ \begin{array}{cc} 1, & 1I, \\ -1, & -1I, \\ i, & iI, \\ -i, & -iI \end{array} \right\}.$$

1. Define $*: G_2^t[I] \times G_2^t[I] \longmapsto G_2^t[I]$ such that

$$\begin{aligned} x * y &= (x_1 + x_2 I) * (y_1 + y_2 I) \\ &= (x_1 + y_1) + (x_2 + y_2) I, \end{aligned}$$

   $\forall\ x, y \in G_2^t[I]$, where $+$ is an addition of complex numbers. $*$ is not a neutrosophic binary operation on $G_2^t[I]$, because if i and $iI \in G_2^t[I]$, then $i + iI \notin I$. Hence $*$ is not a neutrosophic binary operation on $G_2^t[I]$.

2. Define $*: G_2^t[I] \times G_2^t[I] \longmapsto G_2^t[I]$ such that

$$x * y = \begin{cases} xy, \text{if } x \text{ is not indetermenc and } y \\ \quad \text{is indetermenc} \\ (xy) I, \text{if } x \text{ and } y \text{ are an indetermenc} \\ (xy) I^2, \text{if } x \text{ and } y \text{ are not indetermenc} \end{cases}$$

   $\bullet$ denotes multiplication of complex numbers. $*$ is a neutrosophic binary operation on $G_2^t[I]$ as shown in Table(8).

Table 8. for the neutrosophic binary operation on $G_2^t[I]$.

*	1	−1	i	$-i$	I	$-I$	iI	$-iI$
1	1	−1	i	$-i$	I	$-I$	iI	$-iI$
−1	−1	1	$-i$	i	$-I$	I	$-iI$	iI
i	i	$-i$	−1	1	iI	$-iI$	$-I$	I
$-i$	$-i$	i	1	−1	$-iI$	iI	I	$-I$
I	$1I$	$-1I$	iI	$-iI$	1	$-I$	iI	$-iI$
$-I$	$-1I$	$1I$	$-iI$	iI	$-I$	I	$-iI$	iI
iI	iI	$-iI$	$-I$	I	iI	$-iI$	$-I$	I
$-iI$	$-iI$	iI	I	$-I$	$-iI$	iI	I	$-I$

We see from Table(8) that there is a neutrosophic identity element $e_N = 1$. Every neutrosophic element in $G_3^t[I]$ has no inverse; therefore, $*$ has no neutrosophic elements. Hence $(G_2^t[I], *)$ is not a neutrosophic group.

Example 2.5 [8] Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be a classical set and the neutrosophic set of type-1 is given by:

$$\mathbb{Z}_{6\,1}^{\ t}[I] = \left\{ \begin{array}{c} 0, I, 2I, 3I, 4I, 5I, \\ 1, 1+I, 1+2I, 1+3I, 1+4I, 1+5I, \\ 2, 2+I, 2+2I, 2+3I, 2+4I, 2+5I, \\ 3, 3+I, 3+2I, 3+3I, 3+4I, 3+5I, \\ 4, 4+I, 4+2I, 4+3I, 4+4I, 4+5I, \\ 5, 5+I, 5+2I, 5+3I, 5+4I, 5+5I, \end{array} \right\}$$

and the order of $\psi\left(\mathbb{Z}_{6\,1}^{\ t}[I]\right) = 36$.

$N(\mathbb{Z}_6) = \left\langle \mathbb{Z}_{6\,1}^{\ t}[I], \oplus_6 \right\rangle$ be a finite neutrosophic group generated by I and $\mathbb{Z}_6$ under addition modulo 6. Consider $H_1^t[I] = \{0, 3, 3I, 3+3I\}$, where $H = \{0, 3\}$. Since $H_1^t[I]$ is finite, as shown in Table(9). $H_1^t[I] \precnsim_N \mathbb{Z}_{6\,1}^{\ t}[I]$.

Table 9. of $H_1^t[I] \precnsim_N \mathbb{Z}_{6\,1}^{\ t}[I]$.

$\oplus_6$	0	3	$3I$	$3+3I$
0	0	3	$3I$	$3+3I$
3	3	0	$3+3I$	$3I$
$3I$	$3I$	$3+3I$	0	3
$3+3I$	$3+3I$	$3I$	3	0

Note that the neutrosophic order $\psi\left(H_1^t[I]\right) = 4$, $\psi\left(\mathbb{Z}_{6\,1}^{\ t}[I]\right) = 36$, and $\psi\left(H_1^t[I]\right)$ is divides $\psi\left(\mathbb{Z}_{6\,1}^{\ t}[I]\right) = 36$.

Example 2.6 Let $Z_3 \backslash \{0\} = \{1, 2\}$ be a classical set and $Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\} = \left\{ \begin{array}{c} 1+I, 1+2I, \\ 2+1I, 2+2I \end{array} \right\}$ be a neutrosophic set of type 1. Define a neutrosophic binary operation $*: Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\} \times Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\} \longmapsto Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\}$ such that $x*y = (x_1 + x_2 I) * (y_1 + y_2 I) = (x_1 \bullet y_1) + (x_2 \bullet y_2) I)$, where $\bullet$ is a multiplication of mod 3 on $Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\}$ as shown in Table(10).

Table 10. for the neutrosophic binary operation $*$.

$\otimes_3$	$1+I$	$1+2I$	$2+I$	$2+2I$
$1+I$	$1+I$	$1+2I$	$2+I$	$2+2I$
$1+2I$	$1+2I$	$1+I$	$2+2I$	$2+I$
$2+I$	$2+I$	$2+2I$	$1+I$	$1+2I$
$2+2I$	$2+2I$	$2+I$	$1+2I$	$1+I$

It is clear that $*$ is a closure, and it has a neutrosophic identity element $e_N = 1+I$. Furthermore, every neutrosophic element has an inverse, as shown in the following table.

Table 11. for the neutrosophic binary operation $*$.

$(g_1 + g_2 I)$	$1+I$	$1+2I$	$2+I$	$2+2I$
$(g_1 + g_2 I)^{-1}$	$1+I$	$1+2I$	$2+I$	$2+2I$

Hence $N(G) = \left(Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\}, *\right)$ is an abelian neutrosophic group.

Note that. If $*: Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\} \times Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\} \longmapsto Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\}$ such that

$$\begin{aligned} x * y &= (x_1 + x_2 I) * (y_1 + y_2 I) \\ &= (x_1 \bullet y_1) + (x_1 \bullet y_2 + x_2 y_1 + x_2 \bullet y_2) I, \end{aligned}$$

$\forall\ x, y \in Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\}$, where $\bullet$ is a multiplication of mod 3 on $Z_{3\,1}^{\ t}[\mathrm{I}] \backslash \{0\}$. We see that from Table(12).

Table 12. for neutrosophic operation.

$\otimes_3$	$1+I$	$1+2I$	$2+I$	$2+2I$
$1+I$	1	$1+2I$	$2+I$	2
$1+2I$	$1+2I$	$1+2I$	$2+I$	$2+I$
$2+I$	$2+I$	$2+I$	$1+2I$	$1+2I$
$2+2I$	2	$2+I$	$1+2I$	1

$*$ is not a neutrosophic binary operation, because it is not closure.

Definition 2.4 Let $N(G) = (G_1^t[I], *)$ be a neutrosophic group and $N(H) = (H_1^t[I\ ], *)$ be a neutrosophic subgroup of $NG = (G_1^t[I\ ], *)$. If $x, y \in G_1^t[I\ ]$. We said that x is neutrosophic congruent to y module $H_1^t[I\ ]$, if $xy^{-1} \in H_1^t[I\ ]$. By symbolization, $xy^{-1} \in H_1^t[I\ ] \Leftrightarrow x \equiv y \pmod{H_1^t[I\ ]}$.

Theorem 2.5 The neutrosophic congruent relation is a neutrosophic equivalence relation.

Proof.

1. Since $H_1^t[I\ ] \preccurlyeq_N G_1^t[I]$, we have $e_N = xx^{-1}$. This implies that $x \equiv x \pmod{H_1^t[I\ ]}$. Thus $\equiv$ is a neutrosophic reflexive relation.
2. Suppose that $x \equiv y \pmod{H_1^t[I\ ]}$

$$\Rightarrow\ = xy^{-1} \in H_1^t[I\ ],$$
$$\Rightarrow\ (x_1 + x_2 I)(y_1 + y_2 I)^{-1} \in H_1^t[I\ ],$$
$$\Rightarrow\ \left((x_1 + x_2 I)(y_1 + y_2 I)^{-1}\right)^{-1} \in H_1^t[I],$$

since $H_1^t[I] \preccurlyeq_N G_1^t[I]$,

$$\Rightarrow\ \left((y_1 + y_2 I)^{-1}\right)^{-1} (x_1 + x_2) I^{-1} \in H_1^t[I\ ].$$

By Theorem 2.3, part 2 in [8],

$$\Rightarrow\ (y_1 + y_2 I)\ (x_1 + x_2 I)^{-1} \in H_1^t[I\ ].$$

By Theorem 2.3, part 1 in [8],

$$\Rightarrow\ y\ x^{-1} \in H_1^t[I\ ],$$
$$\Rightarrow\ y \equiv x\ (mod\ H_1^t[I\ ]).$$

Hence $\equiv$ is a neutrosophic symmetric relation.

3. Suppose that $x \equiv y \pmod{H_1^t[I\ ]} \wedge\ y \equiv z \pmod{H_1^t[I\ ]}$,

$\because x \equiv y \pmod{H_1^t[I\ ]}$

$$\Rightarrow xy^{-1} \in H_1^t[I\ ] \Rightarrow (x_1 + x_2 I)(y_1 + y_2 I)^{-1} \in H_1^t[I\ ],$$

$\because y \equiv z \pmod{H_1^t[I\ ]} \Rightarrow yz^{-1} \in H_1^t[I\ ]$
$\Rightarrow (y_1 + y_2 I)(z_1 + z_2 I)^{-1} \in H_1^t[I\ ]$,

$$\Rightarrow \left((x_1 + x_2 I)(y_1 + y_2 I)^{-1}\right)((y_1 + y_2 I)\ (z_1 + z_2 I)^{-1}) \in H_1^t[I],$$

since $H_1^t[I\ ] \preccurlyeq_N G_1^t[I] \Rightarrow$
$\left((x_1 + x_2 I)(y_1 + y_2 I)^{-1}(y_1 + y_2 I)\right)(z_1 + z_2 I)^{-1} \in H_1^t[I\ ]$,

$$\Rightarrow\ ((x_1 + x_2)\ \ (e_1 + e_2 I))(z_1 + z_2)^{-1} \in H_1^t[I],$$
$$\Rightarrow\ (x_1 + x_2)(z_1 + z_2)^{-1} \in H_1^t[I],$$
$$\Rightarrow\ xz^{-1} \in H_1^t[I],$$
$$\Rightarrow\ x \equiv z \left(\mathrm{mod}\ H_1^t[I\ ]\right).$$

Therefore, $\equiv$ is a neutrosophic transitive relation, and consequently, $\equiv$ is a neutrosophic equivalence relation.

Theorem 2.6 If $H_1^t[I\ ] \preccurlyeq G_1^t[I\ ]$ and $M_1^t[I\ ] \preccurlyeq G_1^t[I\ ]$, then $H_1^t[I\ ]M_1^t[I\ ] \preccurlyeq G_1^t[I\ ]$ **iff**

$$H_1^t[I\ ]M_1^t[I\ ] = \left\langle H_1^t[I\ ] \cup M_1^t[I\ ]\right\rangle.$$

Proof. Suppose that $H_1^t[I\ ]M_1^t[I\ ] \preccurlyeq G_1^t[I\ ]$. Let $x \in H_1^t[I\ ]M_1^t[I\ ]$
$\Longrightarrow \exists h \in H_1^t[I\ ] \wedge m \in M_1^t[I\ ]$ such that $x = hm$. Hence $x \in H_1^t[I] \cup M_1^t[I]$, so

$$H_1^t[I\ ]M_1^t[I\ ] \subseteq \left\langle H_1^t[I\ ] \cup M_1^t[I\ ]\right\rangle.$$

Also, if $h \in H_1^t[I] \Longrightarrow \exists e_N$ such that $h = he_N \Longrightarrow H_1^t[I] \subseteq H_1^t[I] \cup M_1^t[I]$, and if $m \in M_1^t[I] \Longrightarrow \exists e_N$ such that $m = me_N \Longrightarrow M_1^t[I] \subseteq H_1^t[I] \cup M_1^t[I]$.
We conclude that
$H_1^t[I\ ]M_1^t[I\ ] = \left\langle H_1^t[I] \cup M_1^t[I]\right\rangle$. Conversely, suppose that $H_1^t[I]M_1^t[I] = \left\langle H_1^t[I] \cup M_1^t[I]\right\rangle$. Let $x, y \in H_1^t[I]M_1^t[I]$, since $x \in H_1^t[I]M_1^t[I]$
$\Longrightarrow \exists\left(h \in H_1^t[I]\right) \wedge \exists\left(m \in M_1^t[I]\right)$ such that $x = hm$
$\Longrightarrow \left(h^{-1} \in H_1^t[I]\right) \wedge \left(m^{-1} \in M_1^t[I]\right)$ for some $y^{-1} = h^{-1}m^{-1} = (mh)^{-1}$, hence $xy^{-1} = (hm)(mh)^{-1} \in H_1^t[I]M_1^t[I] \preccurlyeq G_1^t[I]$.

3. Neutrosophic Cosets and Their Properties

In this section, we present the neutrosophic left/right cosets with their properties.

Theorem 3.1 Let $N(H) = \left\langle H_1^t[I\ ], *\right\rangle$ be a neutrosophic subgroup of a neutrosophic group $N(G) = \left\langle G_1^t[I], *\right\rangle$, and $x, y \in G_1^t[I]$. Define a neutrosophic relation $\mathcal{R}$ on the neutrosophic group $G_1^t[I]$ such as $x\mathcal{R}y \Leftrightarrow xy^{-1} \in H_1^t[I]$, $\forall\ x, y \in G_1^t[I]$, then $\mathcal{R}$ is a neutrosophic equivalence relation on $G_1^t[I]$.

Proof. Suppose that $x, y, z \in G_1^t[I]$.

1. Since, $xx^{-1} = (x_1 + x_2 I)(x_1 + x_2 I)^{-1}$

$$= (x_1 + x_2 I)(x_1{}^{-1} + x_2{}^{-1} I)$$
$$= (x_1 x_1{}^{-1} + x_2 x_2{}^{-1} I)$$
$$= (e_1 + e_2 I) = e_N \in H_1^t[I].$$

Hence, $\mathcal{R}$ is a neutrosophic reflexive relation.

2. Assume that $x\mathcal{R}y \Longrightarrow$
$xy^{-1} = (x_1 + x_2 I)(y_1 + y_2 I)^{-1} \in H_1^t[I]$

$$(xy^{-1})^{-1} = \left((x_1 + x_2 I)(y_1 + y_2 I)^{-1}\right)^{-1}$$

$$\Longrightarrow (y^{-1})^{-1}\ \ x^{-1} = \left((y_1 + y_2 I)^{-1}\right)^{-1}(x_1 + x_2 I)^{-1}$$

$$\Longrightarrow y\ \ x^{-1} = \left(y_1{}^{-1} + y_2{}^{-1} I\right)^{-1}(x_1 + x_2 I)^{-1}$$

$$= (y_1 + y_2 I)(x_1 + x_2 I)^{-1} \in H_1^t[I].$$

Hence, $\mathcal{R}$ is a neutrosophic relation.

3. Suppose that $x\mathcal{R}y$ and $y\mathcal{R}z$. Since,

$x\mathcal{R}y \Longrightarrow xy^{-1} = (x_1 + x_2 I)(y_1 + y_2 I)^{-1} \in H_1^t[I]$, and

$$y\mathcal{R}z \Longrightarrow yz^{-1} = (y_1 + y_2 I)(z_1 + z_2 I)^{-1} \in H_1^t[I].$$

$$\Longrightarrow (xy^{-1})(yz^{-1}) = \left((x_1 + x_2 I)(y_1 + y_2 I)^{-1}\right)\left((y_1 + y_2 I)(z_1 + z_2 I)^{-1}\right).$$

$$\Longrightarrow (xy^{-1}y)z^{-1} = \left((x_1 + x_2 I)(y_1 + y_2 I)^{-1}(y_1 + y_2 I)\right)(z_1 + z_2 I)^{-1}.$$

$$\Longrightarrow (xe)z^{-1} = ((x_1 + x_2 I)(e_1 + e_2 I))(z_1 + z_2 I)^{-1}.$$

$$\Longrightarrow xz^{-1} = (x_1 + x_2 I)(z_1 + z_2 I)^{-1} \in H_1^t[I] \Longrightarrow x\mathcal{R}z.$$

Hence $\mathcal{R}$ is a neutrosophic transitive relation. Thus $\mathcal{R}$ is a neutrosophic equivalence relation on $G_1^t[I]$. A Description of the neutrosophic class of a neutrosophic element $x \in G_1^t[I]$. have,

$$\overline{x} = [x] = \left\{y \in G_1^t[I] : y\mathcal{R}x\right\}$$

$$= \left\{y \in G_1^t[I] : yx^{-1} = (y_1 + y_2 I)(x_1 + x_2 I)^{-1} \in H_1^t[I]\right\}$$

$$= \{y \in G_1^t[I] : yx^{-1} = h \Leftrightarrow (y_1 + y_2 I)(x_1 + x_2 I)^{-1} = ((h_1 + hy_2 I)), h \in H_1^t[I]\}$$

$$= \{y \in G_1^t[I] : y = hx \Leftrightarrow (y_1 + y_2 I) = ((h_1 + hy_2 I)(x_1 + x_2 I)), h \in H_1^t[I]\}$$

$= H_1^t[I]x$. $\mathcal{R}$ is called a neutrosophic right relation on $G_1^t[I]$ and $H_1^t[I]x$ is a neutrosophic right-cosets of $H_1^t[I]$ in $G_1^t[I]$ containing x. The set of all neutrosophic right-cosets of $H_1^t[I]$ in $G_1^t[I]$ denoted by $\lfloor G_1^t[I]/H_1^t[I\ ]\rfloor_{\mathcal{R}} = \left\{y \in G_1^t[I] : y\mathcal{R}x\right\}$, this neutrosophic set is called a neutrosophic quotient set. Likewise, the neutrosophic relation is defined as $x\mathcal{L}y \Leftrightarrow x^{-1}y \in H_1^t[I], \forall x, y \in G_1^t[I]$. $\mathcal{L}$ is called a neutrosophic left relation and $xH_1^t[I]$ is called the set of all neutrosophic left-cosets of $H_1^t[I\,]$ in $G_1^t[I]$ containing x, denoted by: $\lceil G_1^t[I]/H_1^t[I\,]\rceil_{\mathcal{L}} = \left\{y \in G_1^t[I] : y\mathcal{L}x\right\}$.

Corollary 3.2 If $N(G) = \langle G_1^t[I], *\rangle$ is a commutative (or abelian) neutrosophic group, then the neutrosophic right relation is equal to the neutrosophic left relation, and $e_N H_1^t[I\ ] = H_1^t[I\ ]$.

Proof. Suppose that $x, y \in G_1^t[I]$ such that

$$x\mathcal{R}y \Leftrightarrow xy^{-1} = (x_1 + x_2 I)(y_1 + y_2 I)^{-1} \in H_1^t[I]$$

$$y^{-1}x = (y_1 + y_2 I)^{-1} \in H_1^t[I]$$

$\Leftrightarrow y\mathcal{L}x \Leftrightarrow x\mathcal{L}y$. Thus $\mathcal{R} = \mathcal{L}$. Also, $e_N H_1^t[I\ ] = \{e_N h : h \in H_1^t[I\ ]\}$

$$= \{(e_N + e_N I) * (h_1 + h_2 I) : e_N, h_1, h_2 \in H, \text{ and indeterminacy } I\}$$

$$= \{(e_N h_1) + (e_N h_2) I : e_N, h_1, h_2 \in H, \text{ and indeterminacy } I\}$$

$$= \{h : h \in H_1^t[I\ ]\}$$

$$= H_1^t[I\ ].$$

Theorem 3.3 Let $N(H) = \langle H_1^t[I\ ], *\rangle$ be a neutrosophic subgroup of a neutrosophic group $N(G) = \langle G_1^t[I], *\rangle$, then there is a one-to-one correspondence between a neutrosophic left (right) coset and a neutrosophic right-coset of $H_1^t[I]$ in $G_1^t[I]$.

Proof. Let $L = \left\{xH_1^t[I] : x \in G_1^t[I]\right\}$ and $\mathcal{R} = \left\{H_1^t[I]x : x \in G_1^t[I]\right\}$ be two neutrosophic left-coset, and neutrosophic right-coset respectively. Define $f : \mathcal{L} \longrightarrow \mathcal{R}$ such that $f\left(xH_1^t[I]\right) = H_1^t[I]x^{-1}$, $\forall xH_1^t[I] \in \mathcal{L}$. Assume that $xH_1^t[I\ ], yH_1^t[I\ ] \in \mathcal{L}$ such that $xH_1^t[I] = yH_1^t[I]$, As $xH_1^t[I] = yH_1^t[I] \Rightarrow \exists x_1, x_2, y_1, y_2 \in G$ such that

$$(x_1 + x_2 I)H_1^t[I] = (y_1 + y_2 I)H_1^t[I]$$

$$\Rightarrow (y_1 + y_2 I)^{-1}(x_1 + x_2 I) \in H_1^t[I],$$

$$\Rightarrow (y_1 + y_2 I)^{-1}(x_1 + x_2 I) = \left(y_1^{-1} + y_2^{-1} I\right)\left(x_1^{-1} + x_2^{-1} I\right)^{-1} \in H_1^t[I],$$

$$\Rightarrow H_1^t[I]\left(y_1^{-1} + y_2^{-1} I\right) = H_1^t[I]\left(x_1^{-1} + x_2^{-1} I\right)$$

$$\Rightarrow H_1^t[I]y^{-1} = H_1^t[I]x^{-1} \Rightarrow f\left(yH_1^t[I]\right) = f\left(xH_1^t[I]\right).$$

Suppose that $f\left(xH_1^t[I]\right) = f\left(yH_1^t[I]\right)$

$$\Rightarrow H_1^t[I]x^{-1} = H_1^t[I]y^{-1}$$

$$\Rightarrow x^{-1}\left(y^{-1}\right)^{-1} = (x_1 + x_2 I)^{-1}\left((y_1 + y_2 I)^{-1}\right)^{-1} \in H_1^t[I]$$

$$\Rightarrow (x_1 + x_2 I)^{-1}\left(\left(y_1^{-1}\right)^{-1} + \left(y_2^{-1}\right)^{-1} I\right) \in H_1^t[I]$$

$$\Rightarrow (x_1 + x_2 I)^{-1}(y_1 + y_2 I) \in H_1^t[I]$$

$$\Rightarrow \left((x_1 + x_2 I)^{-1}(y_1 + y_2 I)\right)^{-1} = (y_1 + y_2 I)^{-1}\left((x_1 + x_2 I)^{-1}\right)^{-1} \in H_1^t[I]$$

$$\Rightarrow (y_1 + y_2 I)^{-1}\left(\left(x_1^{-1}\right)^{-1} + \left(x_2^{-1}\right)^{-1} I\right) = (y_1 + y_2 I)^{-1}(x_1 + x_2 I) \in H_1^t[I]$$

$$\Rightarrow (x_1 + x_2 I)H_1^t[I] = (y_1 + y_2 I)H_1^t[I].$$

$$\Rightarrow xH_1^t[I] = yH_1^t[I].$$

Therefore, f is a neutrosophic injective function.

Suppose that $H_1^t[I]x \in \mathcal{R}$. Since $H_1^t[I]x = H_1^t[I]\left(x^{-1}\right)^{-1} = f\left(xH_1^t[I]\right)$. Hence is a neutrosophic surjective function, and consequently, f is a neutrosophic bijective function.

Theorem 3.4 Let $N(H) = \langle H_1^t[I\ ], *\rangle$ be a neutrosophic subgroup of a neutrosophic group $N(G) = \langle G_1^t[I], *\rangle$, then the neutrosophic elements of $H_1^t[I]$ is a one-to-one correspondence with the neutrosophic elements of any neutrosophic left-coset/ a neutrosophic right-coset of $H_1^t[I\ ]$ in $G_1^t[I]$.

Proof. Consider $H_1^t[I] \precnsim_N G_1^t[I]$, $x \in G_1^t[I]$, and $xH_1^t[I]$ is any a neutrosophic left-coset of $H_1^t[I]$ in $G_1^t[I]$. To show that there exists a one-to-one neutrosophic function of $H_1^t[I]$ onto $xH_1^t[I]$. Define a neutrosophic function $f: H_1^t[I\ ] \longrightarrow xH_1^t[I\ ]$

$$f(h) = xh \Leftrightarrow f(h_1 + h_2 I) = (x_1 + x_2 I)(h_1 + h_2 I) \Leftrightarrow$$

$$((x_1 h_1) + (x_2 h_2) I), \ \forall h \in H_1^t[I\ ].$$

Let $h, h' \in H_1^t[I\ ]$ such that $h = h'$; as $h = h' \Rightarrow f(h) = f\left(h'\right)$

$$\Rightarrow xh = xh'$$

$$\Rightarrow (x_1 + x_2 I)(h_1 + h_2 I) = (x_1 + x_2 I)\left(h'_1 + h'_2 I\right)$$

$$\Rightarrow ((x_1 h_1) + (x_2 h_2) I) = \left((x_1 h'_1) + (x_2 h'_2) I\right),$$

$\forall h, h' \in H_1^t[I]$. Hence, f is well-defined.

Suppose that, $f(h) = f\left(h'\right)$

$$\Rightarrow xh = xh'$$
$$\Rightarrow x^{-1}(xh) = x^{-1}\left(xh'\right)$$
$$\Rightarrow \left(x^{-1}x\right)h = \left(x^{-1}x\right)h'$$
$$\Rightarrow \left(\left(x_1{}^{-1} + x_2{}^{-1} I\right)(x_1 + x_2 I)\right)(h_1 + h_2 I)$$
$$= \left(\left(x_1{}^{-1} + x_2{}^{-1} I\right)(x_1 + x_2 I)\right)\left(h'_1 + h'_2 I\right)$$
$$\Rightarrow \left(\left(x_1{}^{-1}x_1 + x_2{}^{-1}x_2\ I\right)\right)\left(h'_1 + h'_2 I\right)$$
$$\Rightarrow \left(\left(\left(e_1{}^{-1}e_1\right) + \left(e_2{}^{-1}e_2\right) I\right)\right)(h_1 + h_2 I)$$
$$= \left(\left(\left(e_1{}^{-1}e_1\right) + \left(e_2{}^{-1}e_2\right) I\right)\right)\left(h'_1 + h'_2 I\right)$$
$$\Rightarrow (e_1 + e_2 I)(h_1 + h_2 I) = (e_1 + e_2 I)\left(h'_1 + h'_2 I\right)$$
$$\Rightarrow (e_1 h_1 + (e_2 h_2) I) = \left(e_1 h'_1 + \left(e_2 h'_2\right) I\right)$$
$$\Rightarrow (h_1 + h_2 I) = \left(h'_1 + h'_2 I\right)$$
$$\Rightarrow h = h'.$$

Therefore, f is a neutrosophic injective function. Finally, assume that $xh \in xH_1^t[I]$, where $x \in G_1^t[I]$ and $h \in H_1^t[I]$. As $h \in H_1^t[I]$, then $f(h) = xh$, thus f is a neutrosophic surjective function, and consequently, f is a neutrosophic bijective function, and the neutrosophic order of

$$\psi\left(H_1^t[I]\right) = \psi\left(xH_1^t[I]\right).$$

Observation. By a similar argument, the neutrosophic elements of $H_1^t[I\ ]$ are a one-to-one correspondence with the neutrosophic right-cosets of $H_1^t[I]$ in $G_1^t[I]$.

Corollary 3.3 Consider $H_1^t[I] \precnsim_N G_1^t[I]$. If $x \in G_1^t[I]$. Then the neutrosophic order of $H_1^t[I]$, $xH_1^t[I]$, and $H_1^t[I]x$ are equals. That is

$$\psi\left(H_1^t[I]\right) = \psi\left(xH_1^t[I]\right) = \psi\left(H_1^t[I]x\right).$$

By Theorem 3.3. As we saw from Theorem 3.1, we can define neutrosophic left-relation/ neutrosophic right-relation on $G_1^t[I]$ with respect to $H_1^t[I]$, therefore, the following definition results directly from it.

Definition 3.1 Let $N(H) = \langle H_1^t[I\ ], *\rangle$ be a neutrosophic subgroup of a neutrosophic group $N(G) = \langle G_1^t[I], , *\rangle$, and $x \in G_1^t[I]$. Then, the sets of the form

1. $xH_1^t[I\ ] = \{xh : h \in H_1^t[I]\}$ is called a neutrosophic left-coset of $H_1^t[I]$ in $G_1^t[I]$, and
2. $H_1^t[I]x = \{hx : h \in H_1^t[I]\}$ is called a neutrosophic right-cosets of $H_1^t[I\ ]$ in $G_1^t[I]$, the neutrosophic-element x is called a representative of $xH_1^t[I]$ and $H_1^t[I]x$.

Theorem 3.5 Let $N(H) = \langle H_1^t[I], *\rangle$ be a neutrosophic subgroup of a neutrosophic group $N(G) = \langle G_1^t[I], *\rangle$, and $x, y \in G_1^t[I]$. Then $xH_1^t[I] = H_1^t[I\ ] \Leftrightarrow x \in H_1^t[I\ ]$, and

$$H_1^t[I\ ]y = H_1^t[I\ ] \Leftrightarrow y \in H_1^t[I\ ].$$

Proof.

Consider $x \in H_1^t[I\ ]$ to show that $xH_1^t[I\ ] = H_1^t[I\ ]$. Suppose that $y \in xH_1^t[I\ ]$.
$\Rightarrow \exists\ h \in H_1^t[I\ ]$ such that $y = xh$
$\Rightarrow xh \in H_1^t[I\ ]$ by Theorem 3.1 in [9],
$\Rightarrow y \in H_1^t[I\ ]$, hence $xH_1^t[I\ ] \subset H_1^t[I\ ]$. Conversely, Suppose that

$$y \in H_1^t[I\ ] \Rightarrow y = e_N y$$

$$\Rightarrow y = \left(xx^{-1}\right) * y$$

$$\Rightarrow y = x\ \left(x^{-1}y\right),$$

since $x^{-1}, y \in H_1^t[I]$ and $H_1^t[I] \precnsim_N G_1^t[I]$

$$\Rightarrow y = x^{-1}y \in H_1^t[I]$$

$\Rightarrow y = x\ \left(x^{-1}y\right) \in xH_1^t[I]$, hence $H_1^t[I] \subset xH_1^t[I]$, and consequently, $H_1^t[I\ ] = xH_1^t[I\ ]$. On the other hand, let $x \in G[I]$, and assume that $xH_1^t[I\ ] = H_1^t[I\ ] \Rightarrow xe_N = x \in xH_1^t[I\ ] = H_1^t[I\ ]$. The second part uses a similar technique.

Theorem 3.6 Let $NH = \langle H_1^t[I\ ], *\rangle$ be a neutrosophic subgroup of a neutrosophic group $N(G) = \langle G[I], *\rangle$, and $x, y \in G_1^t[I]$. Then

$xH_1^t[I\ ] = yH_1^t[I\ ] \Leftrightarrow y^{-1}x \in H_1^t[I\ ]$, and

$$H_1^t[I\ ]x = H_1^t[I\ ]y \Leftrightarrow xy^{-1} \in H_1^t[I\ ].$$

Proof.
Suppose that

$$xH_1^t[I\ ] = yH_1^t[I\ ] \Rightarrow y^{-1}xH_1^t[I\ ] = y^{-1}yH_1^t[I\ ]$$
$$\Rightarrow y^{-1}xH_1^t[I\ ] = y^{-1}yH_1^t[I\ ]$$
$$\Rightarrow y^{-1}xH_1^t[I\ ] = e_N H_1^t[I\ ]$$
$$\Rightarrow y^{-1}xH_1^t[I\ ] = H_1^t[I\ ]$$

$\Rightarrow y^{-1}x \in H_1^t[I\ ]$. Conversely, Suppose that $y^{-1}x \in H_1^t[I\ ] \Rightarrow y^{-1}xH_1^t[I\ ] = H_1^t[I\ ]$ " by theorem 3.2"

$$\Rightarrow yy^{-1}xH_1^t[I\ ] = yH_1^t[I\ ]$$
$$\Rightarrow e_N xH_1^t[I\ ] = yH_1^t[I\ ]$$
$$\Rightarrow xH_1^t[I\ ] = yH_1^t[I\ ].$$

The second part is similar to the argument.

Example 3.1 Consider $H_1^t[I\ ] \precnsim_N G_1^t[I]$, in examples 2.2, then all neutrosophic left-cosets/ neutrosophic right-cosets of $H_1^t[I\ ]$ in $G_1^t[I]$ are given by:

$$(a+aI)H_1^t[I\ ] = \{(a+aI)h : h \in H_1^t[I]\} = \{(a+aI)$$
$$(a+aI)\} = \{a+aI\} = H_1^t[I].$$

$$(a+bI)H_1^t[I\ ] = \{(a+bI)h : h \in H_1^t[I\ ]\},$$

$= \{(a+bI)(a+aI)\}$, and

$$= \{(a+bI)\}.$$

$$(b+aI)H_1^t[I\ ] = \{(b+aI)h : h \in H_1^t[I\ ]\},$$

$= \{(b+aI)(a+aI)\}$, and

$$= \{(b+aI)\}.$$

$$(b+bI)H_1^t[I\ ] = \{(b+bI)h : h \in H_1^t[I\ ]\},$$

$= \{(b+bI)(a+aI)\}$, and

$$= \{(b+bI)\}.$$

Example 3.2 Consider $H_1^t[I\ ] \precnsim_N G_1^t[I\ ]$ in example 2.3, to find some neutrosophic left-cosets and neutrosophic right-cosets of $H_1^t[I\ ]$ in $G_1^t[I]$. We have

$$xH_1^t[I] = \{xh : h \in H_1^t[I]\}$$
$$(1+I)H_1^t[I] = \{(1+I)h : h \in H_1^t[I]\}$$
$$= \begin{Bmatrix} (1+I)(1+I), & (1+I)(1-I), \\ (1+1I)(-1+I), & (1+1I)(-1-I) \end{Bmatrix}$$
$$= \begin{Bmatrix} (1+I), & (1-I), \\ (-1+I), & (-1-I) \end{Bmatrix} = H_1^t[I],$$

because $(1+1I) \in H_1^t[I]$,

$$(1+iI)H_1^t[I\ ] = \{(1+iI)h : h \in H_1^t[I\ ]\}$$
$$= \begin{Bmatrix} (1+iI)(1+I), & (1+iI)(1-I), \\ (1+iI)(1+iI), & (1+iI)(-1-I) \end{Bmatrix}$$
$$= \begin{Bmatrix} (1+iI), & (1-iI), \\ (1-I), & (-1-iI) \end{Bmatrix}$$
$$(1-iI)H_1^t[I\ ] = \{(1-iI)h : h \in H_1^t[I\ ]\},$$
$$= \begin{Bmatrix} (1-iI)(1+I), & (1-iI)(1-I), \\ (1-iI)(1+iI), & (1-iI)(-1-I) \end{Bmatrix},$$
$$= \begin{Bmatrix} (1-iI), & (1+iI), \\ (1+I), & (-1+iI) \end{Bmatrix}.$$
$$H_1^t[I\ ](1-I) = \{h(1-I) : h \in H[I]\},$$
$$= \begin{Bmatrix} (1+I)(1-I), & (1-I)(1-I), \\ (-1+I)(1-I), & (-1-I)(1-I) \end{Bmatrix},$$
$$= \begin{Bmatrix} (1-I), & (1+I), \\ (-1-I), & (-1+I) \end{Bmatrix} = H_1^t[I\ ],$$

since $(1-I) \in H_1^t[I\ ]$.

$$H_1^t[I\ ](1+iI) = \{h(1+iI) : h \in H[I]\},$$
$$= \begin{Bmatrix} (1+I)(1+iI), & (1-I)(1+iI), \\ (-1+I)(1+iI), & (-1-I)(1+iI) \end{Bmatrix},$$
$$= \begin{Bmatrix} (1+iI), & (1-iI), \\ (-1+iI), & (-1-iI) \end{Bmatrix}.$$
$$H_1^t[I\ ](1-iI) = \{h(1-iI) : h \in H[I]\},$$
$$= \begin{Bmatrix} (1+I)(1-iI), & (1-I)(1-iI), \\ (-1+I)(1-iI), & (-1-I)(1-iI) \end{Bmatrix},$$
$$= \begin{Bmatrix} (1-iI), & (1+iI), \\ (-1-iI), & (-1+iI) \end{Bmatrix}.$$

Note that. $\psi\left(H_1^t[I]\right) = 4, \psi\left(xH_1^t[I]\right) = 4$and $\psi\left(H_1^t[I]x\right) = 4.$

Example 3.3 Consider $H_1^t[I\ ] \precnsim_N \mathbb{Z}_1^t[I]$ in example 2.1. Then the following is the neutrosophic left coset when $x = (2+3I)$ is given by:

$$(2+3I) + H_1^t[I\ ] = \{(2+3I) + h : h \in H_1^t[I\ ]\}$$

$$= \left\{ \begin{matrix} (2+3I), (2+6I), 2, (2+9I), (2-3I), \\ \vdots \\ (5+3I), (5+6I), 5, (5+9I), (5-3I), \\ \vdots \\ (8+3I), (8+6I), 8, (8+9I), (8-3I), \\ \vdots \\ ((2+n)+3I), ((2+n)+6I), \\ (2+n), ((2+n)+9I), ((2+n)-3I), \\ \vdots \end{matrix} \right\}$$

Theorem 3.7 Let $N(H) = \langle H_1^t[I\ ], *\rangle$ be a neutrosophic subgroup of a neutrosophic group $N(G) = \langle G_1^t[I], *\rangle$, and $x, y \in G_1^t[I]$. Then either

$xH_1^t[I\ ] = yH_1^t[I]$ or $xH_1^t[I] \cap yH_1^t[I] = \emptyset_1^t[I]$, and either $H_1^t[I]x = H_1^t[I]y$ or $H_1^t[I]x \cap H_1^t[I]y = \emptyset_1^t[I]$.

Proof. Consider $H_1^t[I] \precsim_N G[I])$ and $x, y \in G[I]$. Suppose that $xH_1^t[I] \cap yH_1^t[I] \neq \emptyset_1^t[I] \Rightarrow \exists z \in xH_1^t[I] \wedge z \in yH_1^t[I] \Rightarrow \exists h_1 \in H_1^t[I] \wedge \exists h_2 \in H_1^t[I]$ such that $z = xh_1 \wedge z = yh_2$

$$\Rightarrow xh_1 = yh_2$$
$$\Rightarrow y^{-1}xh_1 = y^{-1}yh_2$$
$$\Rightarrow y^{-1}xh_1 = e_N h_2$$
$$\Rightarrow y^{-1}xh_1 = h_2$$
$$\Rightarrow xH_1^t[I] = yH_1^t[I]"$$

by theorem 3.2". The second is similar. **Definition 3.2** [17] Let $\wp_{\aleph} = \left\{ \underbrace{H_i^t[I]}_{\alpha} : \alpha \in i \right\}$ be a family of neutrosophic subsets of $H_i^t[I]$, for any $i = 1, 2, 3$, we said that $\left\{ \underbrace{H_i^t[I]}_{\alpha} : \alpha \in i \right\}$ is a neutrosophic partition of $H_i^t[I]$.If satisfies the following conditions:

1. $\underbrace{H_i^t[I\ ]}_{\alpha} \neq \emptyset_i^t[I\ ]$, for any $i = 1, 2, 3$, and $\forall \alpha \in i$,
2. For each $\underbrace{H_i^t[I\ ]}_{\alpha}$ and $\underbrace{H_i^t[I\ ]}_{\beta}$, then either $\underbrace{H_i^t[I\ ]}_{\alpha} = \underbrace{H_i^t[I]}_{\beta}$ or $\underbrace{H_i^t[I]}_{\alpha} \cap \underbrace{H_i^t[I]}_{\beta} = \emptyset_i^t[I]$, $\forall i = 1, 2, 3$,
3. $H_i^t[I] = \bigcup \underbrace{H_i^t[I]}_{\alpha \in i}$.

Corollary 3.8 Let $H_1^t[I\ ] \precsim_N G_1^t[I]$. Then $xH_1^t[I] = \{xh : h \in H_1^t[I] \wedge x \in G_1^t[I]\}$ forms a neutrosophic-partition of a neutrosophic group $G_1^t[I]$.

Proof. Let $H_1^t[I]$ be a neutrosophic subgroup of a neutrosophic-group $G_1^t[I]$. Consider the neutrosophic partition-set $\wp_{\aleph} = \left\{ \underbrace{xH_1^t[I]}_{\alpha} : \alpha \in i \right\}$,

$$\wp_{\aleph} = \left\{ \underbrace{xh}_{\alpha} : x \in G_1^t[I] \wedge h \in H_1^t[I], \alpha \in i \right\},$$

where $i = \{1, 2, 3, \ldots\}$ $\wp_{\aleph}$ is the set of all neutrosophic left-cosets of $H_1^t[I]$ in $G_1^t[I]$,
By Theorem 3.3, either $xH_1^t[I] = yH_1^t[I]$ or $xH_1^t[I] \cap yH_1^t[I] = \emptyset$, where $x, y \in G_1^t[I]$. Since $xH_1^t[I] \subset G_1^t[I]$, for all $x \in G_1^t[I]$ therefore $\bigcup \underbrace{xH_i^t[I]}_{\alpha \in i} \subset G_1^t[I]$. On the other hand, if $x \in G_1^t[I]$, then $x \in xH_1^t[I] \subset \bigcup \underbrace{xH_i^t[I]}_{\alpha \in i}$, hence $G_1^t[I] \subset \bigcup \underbrace{xH_i^t[I]}_{\alpha \in i}$, thus, $G_1^t[I] = \bigcup \underbrace{xH_i^t[I]}_{\alpha \in i}$, therefore $\wp_{\aleph}$ is a neutrosophic partition of $G_1^t[I]$.

Example 3.4 Consider all the neutrosophic left cosets of $H_1^t[I\ ]$ in $G_1^t[I]$ in Example 2.2, and according to Corollary 3.8, the neutrosoph

$$\wp_{\aleph} = \left\{ \underbrace{xH_1^t[I\ ]}_{\alpha} : \alpha \in i \right\}, \quad i = \{1, 2, 3, 4\},$$

$$\wp_{\aleph} = \left\{ \underbrace{(a+aI)H_1^t[I\ ]}_{1}, \underbrace{(a+bI)H_1^t[I]}_{2}, \underbrace{(b+aI)H_1^t[I\ ]}_{3}, \underbrace{(b+bI)H_1^t[I\ ]}_{4} \right\},$$

and $\wp_{\aleph} = \left\{ \underbrace{\{(a+aI)\}}_{1}, \underbrace{\{(a+bI)\}}_{2}, \underbrace{\{(b+aI)\}}_{3}, \underbrace{\{(b+bI)\}}_{4} \right\}$.
By definition 3.2, we have,

1. $\underbrace{xH_1^t[I\ ]}_{\alpha} \neq \emptyset_1^t[I\ ]$, for any $\alpha \in i$ and $i = \{1, 2, 3, 4\}$,
2. $\underbrace{xH_1^t[I\ ]}_{\alpha \in i} \cap \underbrace{xH_1^t[I\ ]}_{\beta \in i} \neq \emptyset_1^t[I\ ]$, $\alpha \neq \beta$, and
3. $G_1^t[I\ ] = \bigcup \underbrace{xH_1^t[I\ ]}_{\alpha \in i}$.

Definition 3.3 Let $N(H) = \langle H_1^t[I\ ], *\rangle$ be a neutrosophic subgroup of a neutrosophic group $NG = \langle G_1^t[I], *\rangle$ with $|G_1^t[I]|$ neutrosophic finite. The neutrosophic number of distinct neutrosophic left-cosets / distinct neutrosophic right-cosets, written $[G_1^t[I] : H_1^t[I]]$, of $H_1^t[I]$ in $G_1^t[I]$ is called the index of $H_1^t[I]$ in $G_1^t[I]$. In the next theorem, we prove the Lagrange theorem for neutrosophic sets.
Theorem 3.9 (Lagrange's theorem) Let $N(H) = \langle H_1^t[I\ ], *\rangle$ be a neutrosophic subgroup of a finite neutrosophic group $N(G) = \langle G_1^t[I], *\rangle$, then the neutrosophic order of $H_1^t[I]$ divides the neutrosophic order of $G_1^t[I]$, symbolically, $\psi(G_1^t[I]) = [G_1^t[I] : H_1^t[I]]\, \psi(H_1^t[I])$ or $H_1^t[I] \,|\, G_1^t[I]$.

Proof. From the premise $G_1^t[I]$ is a finite neutrosophic group**,** so the neutrosophic number of neutrosophic-left cosets of $H_1^t[I]$ in $G_1^t[I]$ is finite**.** Consider the neutrosophic elements:

$x_1 = x_{11} + x_{12}I, x_2 = x_{21} + x_{22}I, \ldots, x_n = x_{n1} + x_{n2}I$ such that the neutrosophic set: $\{x_1H_1^t[I], x_2H_1^t[I], \ldots, x_nH_1^t[I]\}$ is the set of all distinct neutrosophic left-coset elements of $H_1^t[I]$ in $G_1^t[I]$**.** Then, by using Corollary 3.2, the set $\{x_1H_1^t[I], x_2H_1^t[I], \ldots, x_nH_1^t[I]\}$ forms a neutrosophic partition of a neutrosophic group $G_1^t[I]$, hence $G_1^t[I] = \bigcup_{i=1}^{n} x_i H_1^t[I]$and $x_i H_1^t[I] \cap x_j H_1^t[I] = \emptyset_1^t[I]$ for all $i \neq j$, where $1 \leq i,j \leq n$, therefore $[G_1^t[I] : H_1^t[I]] = n$, and $\psi(G_1^t[I]) = \psi(x_1 H_1^t[I]) + \cdots + \psi(x_n H_1^t[I])$. So, by using Corollary 3.8, $\psi(H_1^t[I]) = \psi(x_i H_1^t[I])$ for all i, $1 \leq i \leq n$, therefore

$$\psi(G_1^t[I]) = \underbrace{\psi(H_1^t[I]) + \cdots + \psi(H_1^t[I])}_{n-times}$$

$$= n\psi(H_1^t[I])$$

$$= [G_1^t[I] : H_1^t[I]]\, \psi(H_1^t[I]).$$

Thus, the neutrosophic order of $H_1^t[I]$divides the neutrosophic order of $G_1^t[I]$.

4.Neutrosophic Normal Subgroups and Neutrosophic Quotient Groups

In section 3, we saw that a neutrosophic subgroup $H_1^t[I]$ of type-1 of a neutrosophic group $G_1^t[I]$ of type-1 induces two decompositions of $G_1^t[I]$, namely, the first by neutrosophic left-cosets and the second by neutrosophic right-cosets. This means that $G_1^t[I]$ can be expressed as a disjoint union of distinct neutrosophic left/right cosets.

Theorem 4.1 Let $NH = \langle H_1^t[I], * \rangle$ be a neutrosophic subgroup of a neutrosophic group

$N(G) = \langle G[I], * \rangle$. Then, the following neutrosophic propositions are equivalent:

1. $\mathcal{R} = \mathcal{L}$,
2. $xH_1^t[I] = H_1^t[I]x, \ \forall\, x \in G_1^t[I]$,
3. $xH_1^t[I] \subseteq H_1^t[I]x, \ \forall\, x \in G_1^t[I]$ or $H_1^t[I]x \subseteq xH_1^t[I], \forall\, x \in G_1^t[I]$,
4. $xH_1^t[I]x^{-1} \subseteq H_1^t[I], \ \forall\, x \in G_1^t[I]$,
5. $xhx^{-1} \in H_1^t[I], \ \forall\, x \in G_1^t[I], \ h \in H_1^t[I]$
6. $xH_1^t[I]x^{-1} = H_1^t[I], \ \forall\, x \in G_1^t[I]$.

Proof. Assume that $\mathcal{R} = \mathcal{L} \iff xH_1^t[I] = H_1^t[I]x$ " by Theorem 3.1 and Definition 3.1 in the previous section".

$\iff xH_1^t[I] \subseteq H_1^t[I]x$" *by Theorem* 3.4 $\in$ [12]".

Let $x \in G_1^t[I]$.

$\iff xH_1^t[I]x^{-1} \subseteq H_1^t[I]x\,x^{-1}$

$\iff xH_1^t[I]x^{-1} \subseteq H_1^t[I]e_N$

$\iff xH_1^t[I]x^{-1} \subseteq H_1^t[I]$ " *by Corollary* 3.1 *in the previous section*".

$\iff x^{-1}xH_1^t[I]x^{-1} \subseteq x^{-1}H_1^t[I]$

$\iff e_N H_1^t[I]x^{-1} \subseteq x^{-1}H_1^t[I]$

$\iff H_1^t[I]x^{-1} \subseteq x^{-1}H_1^t[I]$

$\iff H_1^t[I]x^{-1}x \subseteq x^{-1}H_1^t[I]x$

$\iff H_1^t[I]e_N \subseteq x^{-1}H_1^t[I]x$

$\iff H_1^t[I] \subseteq x^{-1}H_1^t[I]x = x^{-1}H_1^t[I](x^{-1})^{-1}$

$\iff xH_1^t[I]x^{-1} = H_1^t[I]$.

$\iff xH_1^t[I]x^{-1}x = H_1^t[I]x$

$\iff xH_1^t[I]e_N = H_1^t[I]x$

$\iff xH_1^t[I] = H_1^t[I]x$

$\iff \mathcal{R} = \mathcal{L}$.

According to Theorem 4.1, we use any neutrosophic proposition to define a neutrosophic normal (or neutrosophic invariant) subgroup of $G_1^t[I]$,

Definition 4.1 Let $N(G) = (G_1^t[I], *)$ be a neutrosophic group, and $N(H) = (H_1^t[I], *)$ be a neutrosophic subgroup of $N(G) = (G_1^t[I], *)$. $H_1^t[I]$ is called a neutrosophic normal (or neutrosophic invariant) subgroup of $G_1^t[I]$, and denoted by $H_1^t[I] \trianglerighteq G_1^t[I]$, if $xH_1^t[I] = H_1^t[I]x$, for all $x \in G_1^t[I]$.

Theorem 4.2 If $H \trianglerighteq G$, then $H_1^t[I] \trianglerighteq G_1^t[I]$.

Proof. Suppose that $H \trianglerighteq G$. Let $h \in H_1^t[I]$ and $x \in G_1^t[I]$.

Since, $h \in H_1^t[I] \Longrightarrow \exists h_1, h_2 \in H$, and indeterminacy I such that $h = h_1 + h_2 I$, and so,

Since, $x \in G_1^t[I] \Longrightarrow \exists x_1, x_2 \in G$, and indeterminacy I such that $x = x_1 + x_2 I$.

$\Longrightarrow \exists {x_1}^{-1}, {x_2}^{-1} \in G$, and indeterminacy I such that

$$x^{-1} = {x_1}^{-1} + {x_2}^{-1} I.$$

$\Longrightarrow \exists x_1 h_1 {x_1}^{-1}, x_2 h_2 {x_2}^{-1} \in H$, and indeterminacy I such that

$$x \ \ hx^{-1} = x_1 h_1 {x_1}^{-1} + x_2 h_2 {x_2}^{-1} I \in H_1^t[I].$$

$$\Longrightarrow H_1^t[I] \trianglerighteq G_1^t[I].$$

Observation. In particular, if $N(G) = (G_1^t[I], *)$ is a commutative (or abelian) neutrosophic group, then $xH_1^t[I] = H_1^t[I]x$.

Example 4.1 Let $N(\mathbb{R}) = (\mathbb{R}_1^t[I], +)$ and $N(\mathbb{Z}) = (\mathbb{Z}_1^t[I], +)$ be two neutrosophic groups under the usual neutrosophic addition, then $\mathbb{Z}_1^t[I] \trianglerighteq \mathbb{R}_1^t[I]$. To show that $\mathbb{Z}_1^t[I] \preccurlyeq \mathbb{R}_1^t[I]$. Let $x, y \in \mathbb{Z}_1^t[I]$ $\Longrightarrow \exists x_1, x_2, y_1, y_2 \in \mathbb{Z}$, and indeterminacy I such that $x = x_1 + x_2 I$ and $y = y_1 + y_2 I$

$\Longrightarrow \exists -x_1, -x_2, -y_1, -y_2 \in \mathbb{Z}$, and indeterminacy I such that

$-x = -x_1 - x_2 I$, and$-y = -y_1 - y_2 I$

$\Longrightarrow x - y = (x_1 + x_2 I) + (-y_1 - y_2 I)$, and indeterminacy I

$= (x_1 - y_1) + (x_2 - y_2) I \in \mathbb{Z}_1^t[I]$. Hence $\mathbb{Z}_1^t[I] \preccurlyeq \mathbb{R}_1^t[I]$. Let $x \in \mathbb{Z}_1^t[I]$ and $y \in \mathbb{R}_1^t[I]$.

$\Longrightarrow \exists x_1, x_2 \in \mathbb{Z}$ and $y_1, y_2 \in \mathbb{R}$ indeterminacy I such that $x = x_1 + x_2 I$ and $y = y_1 + y_2 I$

$$\Longrightarrow y + x - y = (y_1 + y_2 I) + (x_1 + x_2 I) - (y_1 + y_2 I)$$

$$= (y_1+x_1) + (x_2+y_2)I - (y_1+y_2I)$$
$$= (y_1+x_1) + (x_2+y_2)I - (y_1+y_2I)$$
$$= ((y_1+x_1) - y_1) + ((x_2+y_2) - y_2)I$$
$$= x_1 + x_2 I \in \mathbb{Z}_1^t[I].$$

Therefore $\mathbb{Z}_1^t[I] \trianglerighteq \mathbb{R}_1^t[I]$.

Theorem 4.3 Let $H_1^t[I] \trianglerighteq G_1^t[I]$ and $M_1^t[I] \trianglerighteq G_1^t[I]$ be two neutrosophic normal subgroups of $G_1^t[I]$. Then $H_1^t[I] \cap M_1^t[I] \trianglerighteq G_1^t[I]$.

Proof. From the premise, we have $(H_1^t[I] \trianglerighteq G_1^t[I]) \wedge (M_1^t[I] \trianglerighteq G_1^t[I]$.
Since, $H_1^t[I] \trianglerighteq G_1^t[I] \implies xhx^{-1} \in H_1^t[I], \forall\ h \in H_1^t[I],\ x \in G_1^t[I]$, also,
Since, $M_1^t[I] \trianglerighteq G_1^t[I] \implies xmx^{-1} \in M_1^t[I], \forall\ m \in M_1^t[I],\ x \in G_1^t[I]$.

$$\implies (xhx^{-1})(xmx^{-1}) = xhmx^{-1} \in H_1^t[I] \cap M_1^t[I],$$
$$\forall\ h \in H_1^t[I],$$
$$m \in M_1^t[I],\ x \in G_1^t[I] \implies H_1^t[I] \cap M_1^t[I] \trianglerighteq G_1^t[I].$$

Theorem 4.4 Let $H_1^t[I] \trianglerighteq G_1^t[I]$ and $M_1^t[I] \trianglerighteq G_1^t[I]$ be two neutrosophic normal subgroups of $G_1^t[I]$. Then $H_1^t[I]\,M_1^t[I] = M_1^t[I]\,H_1^t[I] \trianglerighteq G_1^t[I]$.

Proof. Suppose that $H_1^t[I]\,M_1^t[I] \neq M_1^t[I]\,H_1^t[I] \implies \exists hm \in H_1^t[I]\,M_1^t[I] \wedge\ hm \notin M_1^t[I]\,H_1^t[I]$
Since $hm \in H_1^t[I]\,M_1^t[I] \implies h \in H_1^t[I] \wedge m \in M_1^t[I]$.
On the other hand, Since $hm \notin M_1^t[I]\,H_1^t[I] \implies h \notin H_1^t[I] \wedge m \notin M_1^t[I] \implies (h \in H_1^t[I] \wedge h \notin H_1^t[I]) \wedge (m \in M_1^t[I] \wedge m \notin M_1^t[I])$, this is a contradiction, hence$H_1^t[I]\,M_1^t[I] = M_1^t[I]\,H_1^t[I]$. To show that $H_1^t[I]\,M_1^t[I] \trianglerighteq G_1^t[I]$. Let $x \in G_1^t[I]$.
Since $(H_1^t[I] \trianglerighteq G_1^t[I] \implies xH_1^t[I]\,x^{-1} \subseteq H_1^t[I]) \wedge (M[I] \trianglerighteq G_1^t[I] \implies xM_1^t[I]\,x^{-1} \subseteq M_1^t[I]) \implies xH_1^t[I]\,M_1^t[I]\,x^{-1} = x(xH_1^t[I]\,x^{-1}xM_1^t[I])\,x^{-1} \subseteq H_1^t[I]\,M_1^t[I]$. Therefore $H_1^t[I]\,M_1^t[I] \trianglerighteq G_1^t[I]$ by Pervious theorem 4.1.

Theorem 4.5 Let $H_1^t[I] \trianglerighteq G_1^t[I]$ and $M_1^t[I] \trianglerighteq G_1^t[I]$ be two neutrosophic normal subgroups of $G_1^t[I]$. Then $H_1^t[I\,]\,M_1^t[I\,] = \langle H_1^t[I\,] \cup M_1^t[I\,] \rangle$. **Proof.** Immediately from Theorem2.5.

Theorem 4.6 Let $H_1^t[I]$ be a neutrosophic normal subgroup of $G_1^t[I]$. Consider the set $\lceil G_1^t[I]/H_1^t[I\,] \rceil_{\mathcal{L}} = \{xH_1^t[I] : x \in G_1^t[I]\}$ of all left cosets of $xH_1^t[I]$ in $G_1^t[I]$. Define a neutrosophic binary operation $*$ on $G_1^t[I]/H_1^t[I]$ such that

$$xH_1^t[I] * yH_1^t[I] = x * yH_1^t[I]$$
$$= (x_1+x_2I) * (y_1+y_2I)\,H_1^t[I]$$
$$= (x_1+x_2I) * (y_1+y_2I)\,H_1^t[I]$$
$$= ((x_1*y_1) + (x_2*y_2)\,I),\, H_1^t[I].$$

Then $(\lceil G_1^t[I]/H_1^t[I\,] \rceil_{\mathcal{L}}, *)$ is a neutrosophic group.

Proof. 1. To show that $*$ is well-defined, we note that Suppose that $xH_1^t[I], x'H_1^t[I], yH_1^t[I]$ and $y' \in \lceil G_1^t[I]/H_1^t[I\,] \rceil_{\mathcal{L}}$ such that $(xH_1^t[I],\ yH_1^t[I]) = (x'H_1^t[I], y'H_1^t[I]\ )$.

$$\implies ((xH_1^t[I] = x'H_1^t[I]) \wedge (yH_1^t[I] = y'H_1^t[I])),$$

since $xH_1^t[I] = x'H_1^t[I] \implies \exists h \in H_1^t[I]$ such that

$$x = x'h \iff \exists x_1,\ x_2, x'_1,\ x'_2\ \in G\ ,\ h_1,\ h_2\ \in H$$

and indeterminacy I such that

$$x = x'h \iff (x_1+x_2I) = (x'_1 + x'_2I)(h_1+h_2I)$$
$$= (x'_1h_1) + (x'_2h_2)\,I$$
$$\iff x_1 = x'_1h_1\ \wedge x_2 = x'_2h_2.$$

Since $yH_1^t[I] = y'H_1^t[I] \implies \exists h' \in H_1^t[I]$ such that

$$y = y'h' \iff \exists y_1,\ y_2, y'_1,\ y'_2\ \in G\ ,\ h'_1,\ h'_2\ \in H$$

and indeterminacy I such that

$$y = y'h' \iff (y_1+y_2I) = (y'_1 + y'_2I)(h'_1+h'_2I)$$
$$= (y'_1h'_1) + (y'_2h'_2)\,I$$

$\iff y_1 = y'_1h'_1\ \wedge y_2 = y'_2h'_2$. So,

$$(y'\ x')^{-1}(xy) = ((x'_1+x'_2I)(y'_1+y'_2I))^{-1}((x_1+x_2I)(y_1+y_2I))$$
$$= (y'_1+y'_2I)^{-1}(x'_1+x'_2I)^{-1}\ (x_1+x_2I)(y_1+y_2I))$$
$$= (y'_1+y'_2I)^{-1}(x'_1+x'_2I)^{-1}\ ((x'_1h_1+x'_2h_2I)(y'_1h'_1+y'_2h'_2I))$$
$$= \left(y'^{-1}_1 + y'^{-1}_2 I\right)\left(x'^{-1}_1 + x'^{-1}_2 I\right)((x'_1h_1+x'_2h_2I)(y'_1h'_1+y'_2h'_2I))$$
$$= \left(y'^{-1}_1 + y'^{-1}_2 I\right)\left(x'^{-1}_1 x'_1h_1 + x'^{-1}_2 x'_2h_2I\right)(y'_1h'_1+y'_2h'_2I)$$
$$= \left(y'^{-1}_1 + y'^{-1}_2 I\right)(h_1+h_2I)\ (y'_1h'_1+y'_2h'_2I)$$
$$= \left(y'^{-1}_1 h_1 + y'^{-1}_2 h_2 I\right)\ (y'_1h'_1+y'_2h'_2I)\ \in\ H_1^t[I],$$

and $H_1^t[I] \trianglerighteq G_1^t[I]$. Hence

$(xy)\,H_1^t[I] = (y'\ x')\,H_1^t[I]$. Therefore, $*$ is well defined.

2. Assume that $xH_1^t[I], yH_1^t[I]$, and $zH_1^t[I] \in \lceil G_1^t[I]/H_1^t[I\,] \rceil_{\mathcal{L}}$. We have,

$$(xH_1^t[I]\,yH_1^t[I])\,zH_1^t[I] = ((x_1+x_2I)\,H_1^t[I]\,(y_1+y_2I)\,H_1^t[I]\,(z_1+z_2I)\,H_1^t[I]$$
$$= \left((x_1+x_2I)(y_1+y_2I)\,H_1^t[I]\right)(z_1+z_2I)\,H_1^t[I]$$

$= ((x_1+x_2I)\,(y_1+y_2I))\ \ (z_1+z_2I)\,H_1^t\,[I]$
$= (x_1y_1+ (x_2y_2)\,I)\ \ (z_1+z_2I)\,H_1^t\,[I]$
$= ((x_1y_1)\,z_1+ (x_2y_2)\,z_2I)\,H_1^t\,[I]$
$= (x_1\,(y_1z_1)\,+x_2\,(y_2z_2)\,I)\,H_1^t\,[I]$
$= (x_1+x_2I)\,((y_1z_1) + (y_2z_2)\,I)\,H_1^t\,[I]$
$= (x_1+x_2I)\,((y_1+y_2I)\,(z_1+z_2I))\,H_1^t\,[I]$
$= (x_1+x_2I)\,H_1^t\,[I]\,((y_1+y_2I)\,(z_1+z_2I))\,H_1^t\,[I]$
$= (x_1+x_2I)\,H_1^t\,[I]\,\big((y_1+y_2I)\,H_1^t\,[I]\,(z_1+z_2I)\,H_1^t\,[I]\big)$

$= xH_1^t\,[I]\,\big(\mathrm{y}H_1^t\,[I]\,\mathrm{z}H_1^t\,[I]\big)$. Hence, $*$ is a neutrosophic associative.

3. $\exists e_N H_1^t\,[I] \in \lceil G_1^t\,[I]/H_1^t\,[I\ ]\rceil_{\mathcal{L}}$ such that

$$e_N H_1^t\,[I]\,xH_1^t\,[I] = (e_1+e_2I)\,H_1^t\,[I]\,(x_1+x_2I)\,H_1^t\,[I]$$
$$= (e_1+e_2I)\,(x_1+x_2I)\,H_1^t\,[I]$$
$$= (e_1x_1+ (e_2x_2)\,I)\,H_1^t\,[I]$$
$$= (x_1+x_2I)\,H_1^t\,[I] = xH_1^t\,[I]\,,$$

and
$xH_1^t\,[I]\,e_N H_1^t\,[I] = xH_1^t\,[I]$, for all $xH_1^t\,[I] \in \lceil G_1^t\,[I]/H_1^t\,[I\ ]\rceil_{\mathcal{L}}$.

4. For all $xH_1^t\,[I] \in \lceil G_1^t\,[I]/H_1^t\,[I\ ]\rceil_{\mathcal{L}} \Longrightarrow \exists x^{-1}H_1^t\,[I] \in \lceil G_1^t\,[I]/H_1^t\,[I\ ]\rceil_{\mathcal{L}}$ such that

$$xH_1^t\,[I]\,x^{-1}H_1^t\,[I] = (x_1+x_2I)\,H_1^t\,[I]\,(x_1+x_2I)^{-1}H_1^t\,[I]$$
$$= (x_1+x_2I)\,(x_1+x_2I)^{-1}H_1^t\,[I]$$
$$= (x_1+x_2I)\,\big(x_1{}^{-1}+x_2{}^{-1}I\big)\,H_1^t\,[I]$$
$$= \big(x_1x_1{}^{-1} + x_2x_2{}^{-1}I\big)\,H_1^t\,[I]$$
$$= (e_1+e_2I)\,H_1^t\,[I] = e_N H_1^t\,[I]\,,$$

and $x^{-1}H_1^t\,[I]\,xH_1^t\,[I] = e_N H_1^t\,[I]$.
Hence $\big(\lceil G_1^t\,[I]/H_1^t\,[I\ ]\rceil_{\mathcal{L}}, *\big)$ is a neutrosophic group.

Definition 4.2 Let $N(G) = \big(G_1^t\,[I]\,,*\big)$ be a neutrosophic group, and $N(H) = \big(H_1^t\,[I]\,,*\big)$ be a neutrosophic normal subgroup of $N(G) = \big(G_1^t\,[I]\,,*\big)$. Then $\big(\lceil G_1^t\,[I]/H_1^t\,[I\ ]\rceil_{\mathcal{L}}, *\big)$ is called the neutrosophic quotient group of $G_1^t\,[I]$ by $H_1^t\,[I]$.

5. Conclusion In this study, we examined the neutrosophic left/right cosets, their properties, the neutrosophic largening theorem, neutrosophic normal subgroups, and quotient groups.

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