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Existence, Uniqueness, and Stability Analysis of the Fractional-Order Burke-Shaw Model with ABC-Fractional Derivativ

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Abstract

The Burke-Shaw model (BSM), which is grounded in the Lorenz system, is essential in various areas of physics and engineering. In this paper, we investigate the application of a fractional derivative with a Mittag-Leffler (M-L) type kernel to address the existence, uniqueness, and Hyers-Ulam stability (HUS) of solutions for the fractional-order BSM. We utilize the ABC-fractional derivative, developed by Atangana and Baleanu, as it offers a more adaptable approach suitable for a diverse array of real-world applications. To demonstrate the existence and uniqueness of solutions, as well as HUS, we introduce a set of necessary conditions that ensure the results presented in this study.

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ABC and ABR fractional derivative, ABC fractional integral, fractional Burke-Shaw system, Lipschitz condition, Hyers-Ulam stability

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1. INTRODUCTION

Fractional Calculus is a general subject of applied mathematics which means that it is an extension of derivatives and integrals with integer order to derivatives and integrals with any arbitrary order. Thirty years ago, the paradigm start to shift from pure mathematics to applied mathematics, such that its applications appear in several applied scientific fields, like: engineering, biology, physics, chemistry, viscoelasticity, fluid dynamics, computer science, signal processing, image processing, mechatronics, electrochemistry, etc. For example, see [1–7, 9, 10]. We interest to study the ordinary and partial differential equations with non integer order, because most of models in applied fields in nowadays involve fractional order derivatives and fractional order integrals in their terms and conditions. To learn more information, we refer to see[11–15, 19]. Therefore, large number of researchers studied several aspects of the arbitrary order differential equations. Mathematical tools are extremely useful in modeling of several real processes and phenomena studied in optimal control, mechanics, biology, medicine, biotechnology, economics, electronics, etc. More information about applications in [16–18]. So, first of all, we will present some important contributions of scientists in mathematical models with fractional order derivatives. The authors in [20] used Caputo-Fabrizio derivative to describe a model of the dynamic of hepatitis B virus. Carla M.A. *et al.*[21] analyzed the impact of pre-exposure prophylaxis (PrEP) and screening effects on HIV dynamics in infected patients. Ivo P.[22] described numerical and simulation models for the classical and fractional-order Bloch equations. Khaled M.S.[23] applied Caputo, Caputo-Fabrizio and Atangana-Baleanu in the Liouville-Caputo sense derivatives with a cubic isothermal auto-catalytic chemical model to obtain approximate solutions of this model. Saif U. *et al.*[26] investigated the existence and uniqueness of solution using fixed point Theorem with Atangana-Baleanu derivative for hepatitis B virus model. By using fixed point Theorem, Badr S. TA. *et al.* [27] studied the existence, uniqueness and stability of solution for H1N1 spread model with Atangana–Baleanu fractional derivative. More examples in [24, 25, 28–31]. Recently, Gamal M. *et al.* [33] applied Pyragas method to control the chaotic behavior of the follonimg fractional Burke-Shaw system

$$\begin{aligned} {}^{c}\mathcal{D}_t^{\mu}u(t) &= -\beta\big(u(t)+v(t)\big), \\ {}^{c}\mathcal{D}_t^{\mu}v(t) &= -\big(v(t)+\beta u(t)w(t)\big), \\ {}^{c}\mathcal{D}_t^{\mu}w(t) &= \alpha+\beta v(t)u(t), \end{aligned}$$

where $u, v, w \in \mathbb{R}$ and $\alpha, \beta > 0$. ${}^{c}\mathcal{D}^{\mu}$ is Caputo derivative with order $0 < \mu \leq 1$.

The study of fractional-order systems in the context of the ABC-fractional derivative has gained significant attention in recent years, particularly due to their ability to more accurately describe real-world systems exhibiting memory and hereditary properties. However, despite the growing interest in ABC-fractional-order derivatives, their application to models such as the BSM remains relatively unexplored. For instance, a notable contribution in the literature is the introduction of a fractal-fractional order for the BSM using the Caputo-Fabrizio derivative with an exponential decay kernel [34]. The study demonstrates the existence and uniqueness of the model using fixed-point theory and solves it numerically with a power series method. A novel numerical scheme based on Newton's interpolation polynomial is used to efficiently solve the fractional BSM, highlighting the advantages of fractal-fractional derivatives in capturing complex dynamics in chaotic systems. The authors in [35] compared synchronization times of the BSM using active control and integer- and fractional-order Pecaro-Carroll (P-C) methods. They showed that the optimal fractional-order P-C method synchronizes 2.35 times faster than active control, with an optimal value of 0.1. This faster synchronization reduces communication delays, making the method ideal for secure communication applications, where signals are transferred with minimal delay and near-zero error rates.

While existing literature has examined the BSM using both integer-order and other fractional-order derivatives, there is a notable gap in comprehensive studies that integrate fractional-order derivatives specifically within the framework of the ABC-fractional derivative. Therefore, motivated by the above discussion, the proposed model is formulated as follows:

$$\begin{cases} {}^{ABC}{}_0\mathcal{D}_t^{\mu}u(t) = -\beta\big(u(t)+v(t)\big), \\ {}^{ABC}{}_0\mathcal{D}_t^{\delta}v(t) = -\big(\gamma v(t)+\beta u(t)w(t)\big), \\ {}^{ABC}{}_0\mathcal{D}_t^{\varepsilon}w(t) = \alpha+\beta v(t)u(t), \\ u_0(t)=0, \;\; v_0(t)=0, \;\; w_0(t)=0. \end{cases} \tag{1.1}$$

where $u, v, w \in \mathbb{R}$ and $\alpha, \beta > 0$. ${}^{ABC}{}_0\mathcal{D}_t^{\mu}, {}^{ABC}{}_0\mathcal{D}_t^{\delta}, {}^{ABC}{}_0\mathcal{D}_t^{\varepsilon}$ are Atangana and Baleanu derivatives in Caputo sense with orders $0 < \mu, \delta, \varepsilon \leq 1$. In our knowledge, no one yet has considered the fractional version of BSM with ABC derivative. So, our proposed model is more general and complicated.

This paper aims to fill the mentioned gap by investigating the existence and uniqueness of solutions (EUS), as well as HUS, for the proposed ABC-fractional version of the BSM (1.1). The model considered here involves differential equations with the ABC fractional derivative, which offers a more general and flexible framework for modeling complex physical phenomena. By applying fractional-order calculus to the BSM, we extend the classical Lorenz system into the fractional domain, providing new insights into the stability and behavior of such systems.

Below, we present some key definitions, lemmas, and theorems that will be essential for our study.

Definition 1.1. [36] Fractional ABC derivative in Caputo sense of the function $\psi \in H^*(a,b), b > a, \mu \in [0,1]$ is given by

$$^{ABC}{}_{a}\mathcal{D}^{\mu}_{\tau}\psi(\tau) = \frac{B(\mu)}{1-\mu}\int_a^{\tau} \psi'(s)E_{\mu}[\frac{-\mu(\tau-s)^{\mu}}{1-\mu}]ds, \tag{1.2}$$

where $B(\mu)$ is satisfied the property $B(0) = B(1) = 1$.

Definition 1.2. [32] Fractional ABC derivative in Riemann-Liouville sense of the function $\psi \in H^*(a,b), b > a, \mu \in [0,1]$ is described as follows:

$$^{ABR}{}_{a}\mathcal{D}^{\mu}_{\tau}\psi(\tau) = \frac{B(\mu)}{1-\mu}\frac{d}{d\tau}\int_a^{\tau} \psi(s)E_{\mu}[\frac{-\mu(\tau-s)^{\mu}}{1-\mu}]ds. \tag{1.3}$$

Definition 1.3. [37, 38] Fractional ABC integral of the function $\psi \in H^*(a,b), b > a, 0 < \mu < 1$ is given by

$$^{AB}{}_{a}\mathcal{I}^{\mu}_{\tau}\psi(\tau) = \frac{1-\mu}{B(\mu)}\psi(\tau) + \frac{\mu}{B(\mu)\Gamma(\mu)}\int_a^{\tau} \psi(s)(\tau-s)^{\mu-1}ds. \tag{1.4}$$

Lemma 1.4. [32] The ABC fractional derivative and ABC fractional integral of the function ψ are satisfied Newton-Leibniz formula

$$^{AB}{}_{a}\mathcal{I}^{\mu}_{\tau}\left(^{ABC}{}_{a}\mathcal{D}^{\mu}_{\tau}\psi(\tau)\right) = \psi(\tau) - \psi(a). \tag{1.5}$$

Theorem 1.1. *[32] For two functions ψ, ϕ, the ABC fractional derivative and ABR fractional derivative hold the Lipschitz condition*

$$\|^{ABC}{}_{a}\mathcal{D}^{\mu}_{\tau}\psi(\tau) - {}^{ABC}{}_{a}\mathcal{D}^{\mu}_{\tau}\phi(\tau)\| \le \Lambda\|\psi(\tau) - \phi(\tau)\|, \tag{1.6}$$

$$\|^{ABR}{}_{a}\mathcal{D}^{\mu}_{\tau}\psi(\tau) - {}^{ABR}{}_{a}\mathcal{D}^{\mu}_{\tau}\phi(\tau)\| \le \Lambda\|\psi(\tau) - \phi(\tau)\|. \tag{1.7}$$

The primary contribution of this study is the application of fractional-order derivatives, particularly the ABC-fractional derivative, to the BSM. This novel approach presents a new framework for analyzing the stability and dynamics of systems in physics and engineering, offering a more accurate representation of real-world processes. **Organization of the paper:** The paper is organized into four sections. Section 1 provides a literature review on the BSM, the ABC-fractional derivative, the ABC-fractional integral, and recent developments in fractional calculus, particularly in relation to physical applications. In Section 2, we establish the existence and uniqueness of solutions for the fractional-order BSM using the ABC-fractional derivative. Section 3 is dedicated to demonstrating the HUS. Finally, Section 4 summarizes the key findings of the paper and offers suggestions for future research on the system.

2. EXISTENCE AND UNIQUENESS OF SOLUTION

There is no specific method that provides an exact solution to our system Eq. (1.1). However, under certain conditions, the existence and uniqueness of an exact solution can be ensured. In this section, we investigate the existence and uniqueness of solutions for Eq. (1.1).

By applying the ABC-fractional integral operator to both sides of each equation in Eq. (1.1), the system can be transformed into a Volterra-type integral equation, as shown below:

$$\begin{aligned}
&u(t) - u(0) = \frac{1-\mu}{B(\mu)}\left[-\beta\big(u(t)+v(t)\big)\right] + \\
&\frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left[-\beta\big(u(s)+v(s)\big)\right]ds, \\
&v(t) - v(0) = \frac{1-\delta}{B(\delta)}\left[-\big(\gamma v(t)+\beta u(t)w(t)\big)\right] + \\
&\frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\left[-\big(\gamma v(s)+\beta u(s)w(s)\big)\right]ds, \\
&w(t) - w(0) = \frac{1-\varepsilon}{B(\varepsilon)}\left[\alpha+\beta v(t)u(t)\right] \\
&+\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-s)^{\varepsilon-1}\left[\alpha+\beta v(s)u(s)\right]ds.
\end{aligned} \tag{2.1}$$

For simplicity, we define $F_i, i \in \mathbb{N}_1^3$ as follows:

$$\begin{aligned}
F_1(t,u) &= -\beta\big(u(t)+v(t)\big), \\
F_2(t,v) &= -\big(\gamma v(t)+\beta u(t)w(t)\big), \\
F_3(t,w) &= \alpha+\beta v(t)u(t).
\end{aligned}$$

Theorem 2.1. *The kernels F_1, F_2 and F_3 hold the Lipschitz condition and contractions, If the subsequent respective conditions $0 \le \ell_j < 1, j \in \mathbb{N}_1^3$ are satisfied.*

Proof. First, we assume that u_1, u_2 are functions and a_1, a_2 are non-negative real numbers, such that $\|u_1\| \le a_1, \|u_2\| \le a_2$, then we have

$$\begin{aligned}
\|F_1(t,u_1) - F_1(t,u_2)\| &= \|-\beta\big(u_1(t)-u_2(t)\big)\| \\
&\le \beta\|u_1(t)-u_2(t)\|.
\end{aligned} \tag{2.2}$$

Taking $\ell_1 = \beta$, we obtain

$$\|F_1(t,u_1) - F_1(t,u_2)\| \le \ell_1\|u_1(t)-u_2(t)\|. \tag{2.3}$$

From Eq. (2.3), we find that the kernel F_1 is satisfying the Lipschitz condition, moreover if $0 \le \ell_1 < 1$, then the kernel F_1 is contraction.

Second, we assume that v_1, v_2 are functions and b_1, b_2 are non-negative real numbers, such that $\|v_1\| \le b_1, \|v_2\| \le b_2$, then we have

$$\begin{aligned}\|F_2(t,v_1) - F_2(t,v_2)\| &= \|-\gamma(v_1(t) - v_2(t))\| \\ &\le \gamma\|v_1(t) - v_2(t)\|. \end{aligned} \tag{2.4}$$

Taking $\ell_2 = \gamma$, we get

$$\|F_2(t,v_1) - F_2(t,v_2)\| \le \ell_2\|v_1(t) - v_2(t)\|. \tag{2.5}$$

From Eq. (2.5), we observe that the kernel F_2 satisfies the Lipschitz condition. Furthermore, if $0 \le \ell_2 < 1$, the kernel F_2 becomes a contraction.

Finally, we assume that w_1, w_2 are functions and c_1, c_2 are non-negative real numbers, such that $\|w_1\| \le c_1, \|w_2\| \le c_2$, then we have

$$\|F_3(t,w_1) - F_3(t,w_2)\| = 0 \le \ell_3\|w_1(t) - w_2(t)\|. \tag{2.6}$$

From Eq. (2.6), we find that the kernel F_3 is satisfying the Lipschitz condition, moreover if $0 \le \ell_3 < 1$, then the kernel F_3 is contraction. □

By using the above kernels, one can rewrite the system Eq. (2.1) in the following simple form:

$$\begin{aligned} u(t) &= \frac{1-\mu}{B(\mu)}F_1(t,u) + \\ &\frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}F_1(s,u(s))ds, \\ v(t) &= \frac{1-\delta}{B(\delta)}F_2(t,v) + \\ &\frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}F_2(s,v(s))ds, \\ w(t) &= \frac{1-\varepsilon}{B(\varepsilon)}F_3(t,w) + \\ &\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-s)^{\varepsilon-1}F_3(s,w(s))ds. \end{aligned} \tag{2.7}$$

Now, we construct the subsequent recursive formula as follows:

$$\begin{aligned} u_n(t) &= \frac{1-\mu}{B(\mu)}F_1(t,u_{n-1}) + \\ &\frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}F_1(s,u_{n-1}(s))ds, \\ v_n(t) &= \frac{1-\delta}{B(\delta)}F_2(t,v_{n-1}) + \\ &\frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}F_2(s,v_{n-1}(s))ds, \\ w_n(t) &= \frac{1-\varepsilon}{B(\varepsilon)}F_3(t,w_{n-1}) + \\ &\frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-s)^{\varepsilon-1}F_3(s,w_{n-1}(s))ds. \end{aligned} \tag{2.8}$$

Let us define a new expressions for the difference between the successive term as follows:

$$\begin{aligned} &{}_u\mathfrak{D}_n(t) = u_n(t) - u_{n-1}(t) = \frac{1-\mu}{B(\mu)}(F_1(t,u_{n-1}) \\ &- F_1(t,u_{n-2})) + \frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}(F_1(s,u_{n-1}) \\ &- F_1(s,u_{n-2}(s)))ds, \\ &{}_v\mathfrak{D}_n(t) = v_n(t) - v_{n-1}(t) = \frac{1-\delta}{B(\delta)}(F_2(t,v_{n-1}) \\ &- F_2(t,v_{n-2})) + \frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(F_2(s,v_{n-1}) \\ &- F_2(s,v_{n-2}(s)))ds, \\ &{}_w\mathfrak{D}_n(t) = w_n(t) - w_{n-1}(t) = \frac{1-\varepsilon}{B(\varepsilon)}(F_3(t,w_{n-1}) \\ &- F_3(t,w_{n-2})) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\int_0^t (t-s)^{\varepsilon-1}(F_3(s,w_{n-1}) \\ &- F_3(s,w_{n-2}(s)))ds. \end{aligned} \tag{2.9}$$

It is interesting to note that

$$\begin{aligned} u_n(t) &= \sum_{i=0}^{n} {}_u\mathfrak{D}_i(t), \\ v_n(t) &= \sum_{i=0}^{n} {}_v\mathfrak{D}_i(t), \\ w_n(t) &= \sum_{i=0}^{n} {}_w\mathfrak{D}_i(t). \end{aligned} \tag{2.10}$$

Taking the norm for both sides of Eq. (2.9)

$$\begin{aligned} \|{}_u\mathfrak{D}_n(t)\| &\le \tfrac{1-\mu}{B(\mu)}\|F_1(t,u_{n-1}) - F_1(t,u_{n-2})\| \\ &+ \tfrac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\|F_1(s,u_{n-1}) \\ &- F_1(s,u_{n-2}(s))\|ds \le \tfrac{1-\mu}{B(\mu)}\ell_1\|u_{n-1}(t) - u_{n-2}(t)\| \\ &+ \tfrac{\mu}{B(\mu)\Gamma(\mu)}\ell_1\int_0^t (t-s)^{\mu-1}\|u_{n-1}(s) - u_{n-2}(s)\|ds. \end{aligned}$$

This implies

$$\|_u\mathfrak{D}_n(t)\| \le \frac{1-\mu}{B(\mu)}\ell_1\|_u\mathfrak{D}_{n-1}(t)\|+ \frac{\mu}{B(\mu)\Gamma(\mu)}\ell_1\int_0^t (t-s)^{\mu-1}\|_u\mathfrak{D}_{n-1}(s)\|ds. \tag{2.11}$$

Similarly, we get the following results:

$$\begin{aligned}\|_v\mathfrak{D}_n(t)\| &\le \frac{1-\delta}{B(\delta)}\ell_2\|_v\mathfrak{D}_{n-1}(t)\|+ \frac{\delta}{B(\delta)\Gamma(\delta)}\ell_2\int_0^t (t-s)^{\delta-1}\|_v\mathfrak{D}_{n-1}(s)\|ds,\\ \|_w\mathfrak{D}_n(t)\| &\le \frac{1-\varepsilon}{B(\varepsilon)}\ell_3\|_w\mathfrak{D}_{n-1}(t)\|+ \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)}\ell_3\int_0^t (t-s)^{\varepsilon-1}\|_w\mathfrak{D}_{n-1}(s)\|ds.\end{aligned} \tag{2.12}$$

By using recursive method with Eq. (2.11) and Eq.Eq. (2.12), we get

$$\begin{aligned}\|_u\mathfrak{D}_n(t)\| &\le \|u_n(0)\|\Big[\frac{(1-\mu)}{B(\mu)}+\frac{1}{B(\mu)\Gamma(\mu)}\Big]^n\ell_1^n,\\ \|_v\mathfrak{D}_n(t)\| &\le \|v_n(0)\|\Big[\frac{(1-\delta)}{B(\delta)}+\frac{1}{B(\delta)\Gamma(\delta)}\Big]^n\ell_2^n,\\ \|_w\mathfrak{D}_n(t)\| &\le \|w_n(0)\|\Big[\frac{(1-\varepsilon)}{B(\varepsilon)}+\frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\Big]^n\ell_3^n.\end{aligned} \tag{2.13}$$

Theorem 2.2. *The ABC fractional system* Eq. (1.1) *has a system of solutions if the following restrictions are hold:*

$$\begin{aligned}&\Big(\frac{(1-\mu)}{B(\mu)}+\frac{1}{B(\mu)\Gamma(\mu)}\Big)\ell_1<1,\\ &\Big(\frac{(1-\delta)}{B(\delta)}+\frac{1}{B(\delta)\Gamma(\delta)}\Big)\ell_2<1,\\ &\Big(\frac{(1-\varepsilon)}{B(\varepsilon)}+\frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\Big)\ell_3<1.\end{aligned} \tag{2.14}$$

Proof. From Eq. (2.7) and Eq. (2.8), we assume

$$\begin{aligned}u(t) &= u_n(t),\\ v(t) &= v_n(t),\\ w(t) &= w_n(t).\end{aligned}$$

Let us define $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ as follows

$$\begin{aligned}\mathcal{A}_n(t) &= u(t)-u_n(t),\\ \mathcal{B}_n(t) &= v(t)-v_n(t),\\ \mathcal{C}_n(t) &= w(t)-w_n(t).\end{aligned}$$

Now, we show that $\|\mathcal{A}_n(t)\|\to 0$, $\|\mathcal{B}_n(t)\|\to 0, \|\mathcal{C}_n(t)\|\to 0$ as $n\to\infty$.

$$\begin{aligned}\|\mathcal{A}_n(t)\| &= \|\frac{1-\mu}{B(\mu)}(F_1(t,u)-F_1(t,u_{n-1}))+ \frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}(F_1(s,u)-F_1(s,u_{n-1}(s)))ds\|\\ &\le \frac{1-\mu}{B(\mu)}\|F_1(t,u)-F_1(t,u_{n-1})\|+\frac{\mu}{B(\mu)\Gamma(\mu)} \times\int_0^t (t-s)^{\mu-1}\|F_1(s,u)-F_1(s,u_{n-1}(s))\|ds\\ &\le \frac{1-\mu}{B(\mu)}\ell_1\|u(t)-u_{n-1}(t)\|+\frac{1}{B(\mu)\Gamma(\mu)}\times \|u(t)-u_{n-1}(t)\|.\end{aligned} \tag{2.15}$$

This implies

$$\|\mathcal{A}_n(t)\| \le \Big(\frac{1-\mu}{B(\mu)}+\frac{1}{B(\mu)\Gamma(\mu)}\Big)\ell_1\|u(t)-u_{n-1}(t)\|. \tag{2.16}$$

With help of Eq. (2.13), we obtain

$$\|\mathcal{A}_n(t)\| \le \Big[\frac{1-\mu}{B(\mu)}+\frac{1}{B(\mu)\Gamma(\mu)}\Big]^{n+1}\ell_1^{n+1}a_1. \tag{2.17}$$

From Eq.(2.17), we see that $\|\mathcal{A}_n(t)\|\to 0$ as $n\to\infty$. By following the same approach and previous steps, we find that $\|\mathcal{B}_n(t)\|\to 0, \|\mathcal{C}_n(t)\|\to 0$ as $n\to\infty$ which complete the proof. □

Theorem 2.3. *The system* Eq. (1.1) *has a unique system of solutions if the following conditions are hold :*

$$\begin{aligned}\Big(\frac{(1-\mu)}{B(\mu)}+\frac{1}{B(\mu)\Gamma(\mu)}\Big)\ell_1-1 &< 0,\\ \Big(\frac{(1-\delta)}{B(\delta)}+\frac{1}{B(\delta)\Gamma(\delta)}\Big)\ell_2-1 &< 0,\\ \Big(\frac{(1-\varepsilon)}{B(\varepsilon)}+\frac{1}{B(\varepsilon)\Gamma(\varepsilon)}\Big)\ell_3-1 &< 0.\end{aligned}$$

Proof. We suppose that there is another system of solutions $u^*(t), v^*(t), w^*(t)$ for the system Eq. (1.1) , then we have

$$\begin{aligned}\|u(t)-u^*(t)\| &\le \frac{1-\mu}{B(\mu)}\|F_1(t,u)-F_1(t,u^*)\|+ \frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t \|F_1(s,u)-F_1(s,u^*)\|(t-s)^{\mu-1}ds\\ &\le \frac{1-\mu}{B(\mu)}\ell_1\|u(t)-u^*(t)\|+\frac{1}{B(\mu)\Gamma(\mu)}\ell_1\|u(t)-u^*(t)\|.\end{aligned} \tag{2.18}$$

Making use of Eq. (2.18), we get

$$\|u(t)-u^*(t)\|\Big(\Big(\frac{(1-\mu)}{B(\mu)}+\frac{1}{B(\mu)\Gamma(\mu)}\Big)\ell_1-1\Big)\ge 0. \tag{2.19}$$

Eq.Eq. (2.19) is valid if and only if

$$\|u(t) - u^*(t)\| = 0.$$

This implies

$$u(t) = u^*(t).$$

Repeating the same procedure with $v(t)$ and $w(t)$, we obtain

$$v(t) = v^*(t), w(t) = w^*(t).$$

This proves that the system Eq. (1.1) has a unique system of solutions. □

3. HYERS-ULAM STABILITY

Definition 3.1. The integral equations Eq. (2.7) is Hyers-Ulam stable if there exists non-negative constants $\Delta_i, i \in \mathbb{N}_1^3$ satisfying:

For every $\alpha_i > 0, i \in \mathbb{N}_1^3$, if

$$\begin{aligned} &|u(t) - \frac{1-\mu}{B(\mu)}F_1(t,u) - \frac{\mu}{B(\mu)\Gamma(\mu)} \times \\ &\int_0^t (t-s)^{\mu-1}F_1(s,u(s))ds| \leq \alpha_1, \\ &|v(t) - \frac{1-\delta}{B(\delta)}F_2(t,v) - \frac{\delta}{B(\delta)\Gamma(\delta)} \times \\ &\int_0^t (t-s)^{\delta-1}F_2(s,v(s))ds| \leq \alpha_2, \\ &|w(t) - \frac{1-\varepsilon}{B(\varepsilon)}F_3(t,w) - \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)} \times \\ &\int_0^t (t-s)^{\varepsilon-1}F_3(s,w(s))ds| \leq \alpha_3, \end{aligned} \tag{3.1}$$

there exist $u^*(t), v^*(t), w^*(t)$ are satisfying

$$\begin{aligned} &u^*(t) = +\frac{1-\mu}{B(\mu)}F_1(t,u^*) + \frac{\mu}{B(\mu)\Gamma(\mu)} \times \\ &\int_0^t (t-s)^{\mu-1}F_1(s,u^*(s))ds, \\ &v^*(t) = +\frac{1-\delta}{B(\delta)}F_2(t,v) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \\ &\int_0^t (t-s)^{\delta-1}F_2(s,v^*(s))ds, \\ &w^*(t) = +\frac{1-\varepsilon}{B(\varepsilon)}F_3(t,w^*) + \frac{\varepsilon}{B(\varepsilon)\Gamma(\varepsilon)} \times \\ &\int_0^t (t-s)^{\varepsilon-1}F_3(s,w^*(s))ds, \end{aligned} \tag{3.2}$$

such that

$$\begin{aligned} &\|u(t) - u^*(t)\| \leq \alpha_1\Delta_1, \\ &\|v(t) - v^*(t)\| \leq \alpha_2\Delta_2 \text{ and} \\ &|w(t) - w^*(t)\| \leq \alpha_3\Delta_3. \end{aligned} \tag{3.3}$$

Theorem 3.2. *The ABC fractional version of Burke-Shaw system* Eq. (1.1) *is Hyers-Ulam stable.*

Proof. Using Definition 3.1 and Eq.Eq. (2.7), we let $(u(t), v(t), w(t))$ to be the fact solution of Eq. (2.7) and $(u^*(t), v^*(t), w^*(t))$ to be an approximate solution satisfying Eq. (3.3). Then, we have

$$\begin{aligned} &\|u(t) - u^*(t)\| \leq \frac{1-\mu}{B(\mu)}\|F_1(t,u) - F_1(t,u^*)\| \\ &+ \frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t \|F_1(s,u) - F_1(s,u^*)\|(t-s)^{\mu-1}ds \\ &\leq \frac{1-\mu}{B(\mu)}\ell_1\|u(t) - u^*(t)\| + \frac{1}{B(\mu)\Gamma(\mu)}\ell_1\|u(t) - u^*(t)\| \\ &= (\frac{1-\mu}{B(\mu)} + \frac{1}{B(\mu)\Gamma(\mu)})\ell_1\|u(t) - u^*(t)\|. \end{aligned} \tag{3.4}$$

Similarly, we get

$$\begin{aligned} \|v(t) - v^*(t)\| \leq \quad &(\frac{1-\delta}{B(\delta)} + \frac{1}{B(\delta)\Gamma(\delta)})\ell_2\|v(t) - v^*(t)\|, \\ \|w(t) - w^*(t)\| \leq \quad &(\frac{1-\varepsilon}{B(\varepsilon)} + \frac{1}{B(\varepsilon)\Gamma(\varepsilon)})\ell_3\|w(t) - w^*(t)\|. \end{aligned} \tag{3.5}$$

Hence, by Eq. (3.4), Eq. (3.5) the integral equations Eq. (2.7) are Hyers-Ulam stable. Thus, the ABC fractional version of BSM Eq. (1.1) is Hyers-Ulam stable. □

4. CONCLUSION

In this paper, we have established the existence, uniqueness, and HUS for a fractional-order BSM using the ABC-fractional derivative. To achieve these results, we transformed the fractional-order BSM Eq. (1.1) into an integral system by applying the properties of the ABC fractional integral. We demonstrated that the fractional-order BSM possesses a unique solution and also satisfies HUS. This work extends the classical Lorenz system, which has wide-ranging applications in both physics and engineering. For future research on the model Eq. (1.1), we recommend exploring the multiplicity of solutions and the possibility of non-solutions, utilizing mathematical techniques such as topological degree theory and the upper-lower solution method. Additionally, investigating the numerical solutions of Eq. (1.1) through various numerical methods could provide valuable insights.

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