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Investigating Quantum Entanglement by Using IBM's Quantum Information Science Kit (QISKit)

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Abstract

Quantum entanglement has gained significant attention in recent years owing to its diverse application in quantum informatics. Its nonlocal nature makes it a crucial resource in quantum information, especially in quantum communication and networking. Entanglement plays a vital role in advancing quantum computation and technology.

This study discusses the creation of entangled quantum states using photon pairs and the measurement of entanglement, utilizing the IBM Q Experience—the first industrial initiative to develop universal quantum computers and provide widespread access to these technologies through cloud platforms. The introduction of the Qiskit tool has enabled researchers, educators, developers, and enthusiasts to write code and conduct experiments on quantum machines. In this study, we implemented and tested circuits on the IBM Q quantum computing platform using Qiskit to execute quantum computing programs such as the Bell Inequality, CHSH Inequality, and Five-Qubit GHZ States, with two successful programming presented, these tools, along with the associated Jupyter Notebook pages, serve as additional resources for users of the IBM Q Experience. We present the results which will help evaluate these prototypes' performance. Where quantum computers are designed to process information units that go beyond mere abstract mathematical entities, as suggested by Shannon's theory; instead, they represent real physical objects defined by one of the two fundamental physical theories—quantum mechanics.

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1. INTRODUCTION

The concept of quantum information emerged at the intersection of quantum mechanics and information theory [1]. This concept is deeply intertwined with the mathematical framework of quantum formalism, which establishes essential limitations of the nature of physical laws [2]. In 1935, Albert Einstein, along with Boris Podolsky and Nathan Rosen, introduced a thought experiment to demonstrate the incompleteness of quantum mechanics [3]. They suggested a peculiar interaction between the two particles, Alice and Bob, separated by a significant distance. These particles exist in a combined state with multiple momentum eigenstates, highlighting the strange correlations predicted by quantum theory [4].

This paradox suggests that the global states of a composite system cannot be expressed as products of the states of individual subsystems, as described by quantum mechanics. This phenomenon, referred to as "entanglement," highlights the inherent statistical relationships between subsystems of a quantum system. Entanglement is a pivotal resource in quantum information science, that facilitates key applications such as superdense coding, quantum teleportation, quantum computing, and quantum cryptography [5].

Fundamental Concepts of the Quantum Theory

Applied quantum theory is inherently indeterministic because it focuses on predicting the probabilities of various events rather than providing certainties [6]. In contrast, classical theories, such as those described by Newtonian mechanics, allow precise predictions. Within the Newtonian framework, if the initial positions and velocities of all the interacting objects are known, their trajectories are with complete certainty. This deterministic view of the universe led to a philosophical interpretation of determinism. A notable example is provided by mathematician Pierre-Simon Laplace, who famously posited the idea of a supreme intellect—commonly referred to as “Laplace’s demon.” This hypothetical entity can predict all future events by analyzing the present and past states of the universe [7].

The quantum theory is a foundational framework that describes the behavior of matter and energy at microscopic scales, such as atoms and subatomic particles. Its principles diverge significantly from classical physics, leading to a unique understanding of the physical world [8]: Key Principles

- ***Wave-Particle Duality:*** One of the foundational ideas in quantum theory is wave-particle duality, which asserts that elementary particles, such as electrons, exhibit wave-like and particle-like properties. This means that particles can behave as discrete entities and exhibit interference patterns typical of waves [9].
- ***Superposition of States:*** Quantum systems can exist in multiple states simultaneously. This principle allows a microscopic system to be described as a superposition of different states, meaning that it can be partly in each of them until a measurement is made [10].
- ***Uncertainty Principle:*** Formulated by Werner Heisenberg, this principle states that certain pairs of physical properties, like position and momentum, cannot be simultaneously known to arbitrary precision. The more accurately one property is measured, the less accurately another property can be known. Mathematically, this is expressed as:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

  where Δx is the uncertainty in the position, Δp is the uncertainty in the momentum, and $\hbar$ is the reduced Planck constant.
- ***Indeterminacy:*** Unlike classical theories, which predict exact outcomes given initial conditions, quantum mechanics only provides probabilities for various outcomes. The act of measurement causes a quantum system to "collapse" into one of the possible states, but the specific outcome cannot be predicted with certainty [11].
- ***Complementarity:*** Proposed by Niels Bohr, this concept asserts that different experimental setups reveal different aspects of the quantum phenomena. Observing one property (such as position) affects the measurement of another (such as momentum), emphasizing that measurements are inherently probabilistic and context-dependent.
- ***Measurement Problem:*** The measurement problem in quantum mechanics describes the challenge of understanding how measurements influence quantum systems. When a measurement is performed, the system collapses from a superposition of states to a single outcome; however, the exact mechanism of this transition remains a topic of debate and interpretation [12].
- ***Implications for Understanding Reality:*** The indeterministic nature of quantum mechanics has profound philosophical implications. This challenges classical determinism—the idea that future states can be predicted with certainty based on present knowledge. The Einstein-Podolsky-Rosen (EPR) paradox further illustrates this tension by questioning whether quantum mechanics provides a complete description of physical reality [13].
- ***Entanglement:*** Quantum entanglement refers to a phenomenon in which two or more particles become interconnected such that the state of one particle is directly related to the state of another, regardless of the distance separating them. This unique property allows instantaneous correlations between particles, leading to applications in quantum computing and cryptography [14].

These fundamental concepts of quantum theory form the basis of many innovations in modern physics and technology, including quantum computing, quantum cryptography, and our understanding of atomic structures. By challenging classical notions, they offer profound insights into the nature of reality at the smallest scales [15].

The final and perhaps most remarkable feature of quantum mechanics is entanglement. Unlike classical phenomena, there is no true equivalent of entanglement within classical physics. The closest comparison might be drawn to a secret key shared between the two parties, yet even this analogy falls short. Entanglement describes profound quantum correlations that can exist between two or more particles, exhibiting correlations that surpass classical correlations in a precise manner [16].

The term "entanglement" was first introduced by Erwin Schrödinger in 1935, as he explored some of the peculiar properties and implications associated with this phenomenon. Subsequently, Einstein, Podolsky, and Rosen raised an apparent paradox related to entanglement, which cast doubt on the completeness of quantum theory. They questioned whether the unusual characteristics of entanglement undermined the uncertainty principle and proposed the existence of “local hidden-variable” theories that could potentially account for experimental results [17].

It took approximately 30 years to address this paradox, with John Bell providing a resolution through the introduction of a simple inequality, now known as Bell’s inequality [18]. He demonstrated that classical correlations between two particles, adhering to the assumptions of the local hidden-variable theory proposed by Einstein, Podolsky, and Rosen, must fall below a specific threshold. Conversely, he showed that the correlations observed in two entangled quantum particles can exceed this limit, indicating that entanglement cannot be understood through classical correlations, but is a distinct quantum phenomenon.

Subsequent experiments have validated that pairs of entangled quantum particles can violate Bell's inequality, reinforcing the unique nature of entanglement in quantum mechanics [19].

2. PRELIMINARIES

In quantum mechanics, a quantum system is often represented using capital letters $(e.g., A, B)$ and modeled within finite-dimensional Hilbert spaces, such as H_A and H_B [20]. This representation is crucial for understanding the behavior of quantum states and their interactions, particularly in the context of quantum information theory. The collection of linear operators acting on system A is denoted as $P(A)$, whereas the set of positive semidefinite operators is noted as $L(A)$. The identity operator for system A is denoted as 1_A. The quantum states associated with system A are defined as $S(A) \equiv \rho_A \mid \rho_A = 0, Tr\rho_A = 1.$ A where

- $\rho_A = 0$: This indicates that the state is a valid quantum state if it is not null.
- $Tr\rho_A = 1$: This condition signifies that the trace of the quantum state density operator is equal to one, which is a requirement for normalized quantum states, and a sub-normalized state is characterized as a positive semidefinite operator with a trace that does not exceed one. Additionally, any two Hermitian operators X were considered within this framework.

2.1. The State Space and the Dirac Notation

Quantum systems, including atoms with energy levels, electrons characterized by spin states, and photons with polarization, can be effectively represented within the framework of quantum mechanics. The mathematical description of a quantum state utilizes a notation known as Dirac or bra-ket notation [21].

Postulate: At any given moment$t$, the state of a physical system is represented by an element $|\psi(t)\rangle$ which belongs to a state space denoted as$H$ This notation is fundamental to quantum mechanics, where the state space is generally a complex Hilbert space [22]. This concept was introduced by Paul Dirac in 1939, who used the notation $|\psi(t)\rangle$ to signify an element of state space. The Hilbert space H is characterized as a finite-dimensional vector space over the complex numbers C equipped with an inner product that is represented as $\langle.|.\rangle$. It is crucial to recognize that identifying the state space using the mathematical notion of a vector space has significant implications [23]. This association permits the formation of linear combinations or superpositions of the elements in H, which also qualify as valid quantum states. To mathematically define the inner product within Hilbert space H, the associated dual space is introduced.

(Dual Space):Let H represent a Hilbert space defined over complex number C. The dual space H^* comprises all linear maps from $H \to C$. Elements of the dual space, denoted as $\langle\varphi|$ act on a state vector $|\psi\rangle \in H$ through the mapping $\langle\varphi| : \langle\psi\rangle \mapsto \langle\varphi|\psi\rangle \in C$. This operation yields a complex scalar known as the inner product, represented as $\langle\varphi|\psi\rangle$, which characterizes the relationship between the states $|\psi\rangle, |\varphi\rangle \in H$. Furthermore, the standard inner product in vector spaces over the complex field $C$is a sesquilinear form, adhering to specific linearity properties [24].

$$\begin{aligned}\langle\varphi \mid \alpha\psi_1 + \beta\psi_2\rangle &= \alpha\langle\varphi \mid \psi_1\rangle + \beta\langle\varphi \mid \psi_2\rangle \\ \langle\alpha\varphi_1 + \beta\varphi_2 \mid \psi\rangle &= \alpha^*\langle\varphi_1 \mid \psi\rangle + \beta^*\langle\varphi_2 \mid \psi\rangle\end{aligned} \tag{1}$$

For all states $|\psi_i\rangle, |\psi\rangle \in H$,and dual states $\langle\varphi|, \langle\varphi_i| \in H^*$, with $\alpha, \beta \in C$ where α^* represents the complex conjugate of α. The inner product establishes a norm $\|\cdot\|$ on H. This norm is defined as $\|\varphi\| := \sqrt{\langle\psi \mid \psi\rangle}$ and states satisfying $\|\psi\| = 1$ are referred to as normalized states.

Additionally, the inner product imparts a notion of relative orientation between states, whereby states with $\langle\varphi \mid \psi\rangle = 0$ are said to be orthogonal to one another.

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Orthonormal Basis Let H be a d-dimensional Hilbert space. A collection of linearly independent vectors $\{|b_i\rangle\}_{i=1}^d := B \subset H$ considered a basis for H if every element $|\psi\rangle \in H$ can be expressed as a unique linear combination of the vectors in B, [25] i.e.,

$$|\psi\rangle = \sum_{i=1}^{d} C_i |b_i\rangle, \text{ with } C_i := \langle b_i \mid \psi\rangle \in C \tag{2}$$

Basis B is referred to as orthonormal if its elements have unit length and are mutually orthogonal. This means that for any two basis elements b_i and b_j the inner product satisfies $\langle b_i \mid b_j\rangle = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta, which is equal to one when $j = i$ and zero otherwise [26]. Commonly, the preferred orthonormal basis is known as the computational basis, which consists of elements $\{|K\rangle\}_{k=0}^{d-1}$.Two orthonormal bases $\{|b_i\rangle\}_{i=1}^d, \left\{\left|b'_j\right\rangle\right\}_{i=1}^d$ of H are mutually unbiased if $\left|\left\langle b_i \mid b_j'\right\rangle\right|^2 = \frac{i}{d}$ for all i, j.

2.2. Linear Operators

To describe the manipulation of a quantum state, we introduce the following definition: ***(Linear Operator)***: A linear operator on a Hilbert space H is defined as a linear mapping [27] $M : H \to H, |\psi\rangle \mapsto M|\psi\rangle$.

Quantum Bit

A qubit, which stands for "quantum bit," is the fundamental unit of information in quantum computing. Although it serves a purpose similar to that of a classical bit, it differs significantly in functionality. While a classical bit can only represent two values $(0\ or 1)$, a qubit can be measured in two states, typically denoted [28] as $|0\rangle$ and $|1\rangle$.

In contrast to a classical bit, a qubit can exist in a superposition of these states before measurement, meaning it can simultaneously represent a combination of $|0\rangle$ and $|1\rangle$, each associated with a certain probability. However, when measured, the qubit state collapses to one of these two definite states.

The notion of a qubit is relatively abstract, as it is not directly related to a physical object in everyday experience. Instead, qubits are often represented by artificial atom systems engineered to imitate the atomic behavior. These artificial systems are designed to replicate the properties of two-state systems, while meeting specific requirements, such as minimal energy dissipation and protection from environmental disturbances [29].

A qubit is an essential component of a quantum computer. It can be represented by particles such as photons or the nuclei of specific elements, which form the core of the qubit and dictate its physical characteristics, including superposition, entanglement, and parallelism

In the context of a quantum computer, qubits are manipulated to signify two different spin states: "spin-up" corresponds to 0, while "spin-down" denotes 1. This distinction is vital for the proper encoding of information, particularly when considering the roles of superposition and entanglement for the same qubit.

Under normal room temperature conditions, these particles are in an unstable state and frequently fluctuate between varying energy levels. However, when subjected to much lower temperatures, they tended to stabilize in the spin-down state. To induce a transition between spin states (either spin-up or back to spin-down), an external energy source must be applied to facilitate this change [30].

3. QUANTUM ENTANGLEMENT

Quantum entanglement is a promising feature of composite physical systems. This illustrates a significant departure of quantum physics from classical physics and has evolved into a vital resource for various applications, including quantum teleportation and quantum cryptography. Additionally, entanglement can lead to another intriguing phenomenon known as Bell non-locality. Consequently, the detection and characterization of entanglement in composite quantum systems remain a central yet unresolved challenge in quantum information theory [31].

3.0.1. Correlations in bipartite quantum systems

For a bipartite pure state $|\psi\rangle_{AB}$, the entanglement of formation is characterized by the entropy of one of its subsystems as follows: $E_f(|\psi\rangle_{AB}) = S(\rho_A)$ Here,ρ represents the reduced density matrix for subsystem A, obtained by performing a partial trace over subsystem B. The von Neumann entropy of the quantum state ρ is defined $S(\rho) = -tr(\rho \ln \rho)$ is the von Neumann entropy of the quantum state ρ. For a bipartite mixed stateρ_{AB} the entanglement of the formation is defined as the minimum average entanglement overall potential decompositions of that state. This can be mathematically expressed as follows:

$$E_f(\rho_{AB}) = min \sum_i P_i E_f(|\Psi\rangle_{AB}), \tag{3}$$

where the summation is taken over all possible pure state decompositions of the mixed state given by

$$\rho_{AB} = \sum_i P_i(|\psi_i\rangle_{AB}). \tag{4}$$

For a probability ensemble $\xi = \{P_i, \rho_i\}$ that results in a quantum state ρ where $\rho = \sum_i P_i \rho_i$, the Holevo quantity is defined as:

$$x(\xi) = S(\rho) - \sum_i P_i S(\rho_i). \tag{5}$$

3.1. Quantum entanglement and its measures

Entanglement is a captivating phenomenon in the field of quantum mechanics. A key illustration of this effect is the singlet state in a system consisting of two spin-1/2 particles, which can be described by the following wave function:

$$|\psi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{6}$$

This wave function demonstrates a coherent superposition of qubits and cannot be factored into the individual wave functions of parts of the system. The entanglement characteristic guarantees that measuring the state of one particle will immediately affect the state of the other, regardless of the distance between them. [32].

Quantum entanglement is a captivating phenomenon that has attracted significant attention in both theoretical and experimental research. The entropy associated with a state represented by a reduced (partial or averaged) density matrix has emerged as a crucial and insightful measure of the entanglement. This concept was first explored in 1986 by using black holes. Currently, the entropy of the reduced state is widely utilized across various scientific disciplines, including quantum field theory, solid-state physics, and quantum information theory [33].

3.2. Pure and mixed states

In quantum mechanics, states can be classified as either pure or mixed, each representing a different level of information about a quantum system [34].

3.2.1. Pure state

A pure state is defined as possessing full knowledge of a quantum system. Mathematically, it is expressed as a state vector (or ket)$|\psi\rangle$ within a Hilbert space. This state cannot be decomposed into a mixture of other states, indicating that it is an extreme point within the manifold of the possible states. In the context of density matrices, a pure state can be represented as follows [35]:

$$\rho = |\psi\rangle\langle\psi| \quad with \quad \|\psi\| = 1 \tag{7}$$

where ρ is a one-dimensional projector. Pure states are associated with maximum certainty regarding the properties of the system.

3.2.2. Mixed States

By contrast, a mixed state signifies a statistical ensemble of various pure states, reflecting a lack of complete information about the system. A mixed state is described by a density matrix that does not act as a projector for a single state. Instead, it can be formulated as a convex combination of several pure states. [19]:

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \tag{8}$$

where p_i are the probabilities associated with each pure state $|\psi_i\rangle$ Mixed states are characterized by their density matrices having more than one non-zero eigenvalue, reflecting the uncertainty inherent in the system.

3.3. The Differences

- ***Information Content***: Pure states contain maximal information, whereas mixed states contain less than maximal information [36].
- ***Mathematical Representation***: Pure states are represented by vectors in Hilbert space or one-dimensional projectors in density matrix form, whereas mixed states are represented by density matrices that can be decomposed into mixtures of pure states.
- ***Purity Measure***: The purity of a state is defined as $\lambda = tr(\rho^2)$. For pure states, this measure equals one, whereas, for mixed states, it falls between $\frac{1}{d}$ and one, where d is the dimension of the Hilbert space.

Understanding these distinctions is crucial for analyzing quantum systems and their behavior in various applications, including quantum computing and quantum information theory.

3.3.1. Entropy of entanglement

Entanglement entropy : Entropement entropy, often referred to as ***entanglement entropy***, measures the extent of quantum entanglement between two subsystems within a composite quantum system. A reduced matrix can be obtained for a pure bipartite quantum state, to represent the state of one of the subsystems. The entropy of entanglement is then defined as the von Neumann entropy of the reduced density matrix. A non-zero value for this entropy signifies that the two subsystems are entangled. [37]. For a bipartite pure state $\rho_{AB} = |\Psi\rangle\langle\Psi|$, the entanglement entropy S can be expressed as:

$$S = S\left(\rho_A\right) = -\operatorname{Tr}\left(\rho_A \ln \rho_A\right) \tag{9}$$

where ρ_A is the reduced density matrix obtained by tracing out the other subsystem. Owing to the properties of Schmidt decomposition, the entanglement entropy is independent of which subsystem's reduced density matrix is used.

3.3.2. Density Matrix

A density matrix, also known as a density operator, is a mathematical construct used in quantum mechanics to describe the statistical state of a quantum system. This provides a comprehensive way to represent both pure and mixed states, making it essential for understanding quantum systems that are not in a definite state. The density matrix ρ is mathematically defined as a positive semi-definite operator with a trace of one for normalized states. The key characteristics of the density matrix are as follows:

- Representation: A density matrix can be represented in a finite-dimensional Hilbert space and is denoted as ρ.
- Normalization: For the pure state, the density matrix satisfies $(\rho) = 1$. This condition ensures that the total probability of finding a system in any of its quantum states is summing to one.
- Mixed States: The density matrix can also describe mixed states, which are statistical mixtures of different quantum states rather than coherent superpositions. Mixed states result from classical uncertainty or lack of complete knowledge of the system.

For a finite-dimensional quantum system a pure state can be expressed as$|\psi\rangle$, leading to the density matrix defined as:

$$\rho = |\Psi\rangle\langle\Psi| \tag{10}$$

This implies that the density matrix is the outer product of the state vector. For a mixed state represented by a probability distribution over different pure states, $|\psi_i\rangle$

with probability P_i:

$$\rho = \sum_i p_i |\Psi\rangle\langle\Psi| \tag{11}$$

Here, the probabilitiesp_i satisfy$\sum_i p_i = 1$ and $p_i \geq 0$

3.3.3. Renyi Entropy

In addition to von Neumann entropy, ***Renyi entropy*** can also be defined for measuring entanglement. The Renyi entanglement entropy S_α for a reduced density matrix ρ_A is given [38] by:

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \ln\left(\mathrm{Tr}\left(\rho_A^\alpha\right)\right) \tag{12}$$

for any index $\alpha \geq 0$.

3.3.4. Area Law

Entanglement entropy often follows an ***area law***, in which the leading term grows proportionally to the boundary area between the two partitions of a system. This behavior is particularly common in the ground states of local gapped quantum many-body systems, significantly simplifying the analysis of such systems [39].

3.4. Separable and Non-Separable States

In quantum mechanics, states can be classified as separable or non-separable (entangled), which reflects the nature of their correlations.

3.4.1. Separable States

A separable state can be expressed as a convex combination of the product states. This means that it can be written in the form [40]:

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i, \tag{13}$$

Where p_i is the probability, and ρ_A^i and ρ_B^i are the states of subsystems A and B, respectively. In simpler terms, separable states exhibit no quantum entanglement; any correlations present can be attributed to classical randomness. For pure states, a state is separable if it can be represented as a product of individual states:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \tag{14}$$

3.4.2. Non-Separable States (Entangled States)

A non-separable state, or entangled state, cannot be expressed as a product of the states of its subsystems. This implies that the quantum state of the system cannot be described independently for each subsystem. Entangled states exhibit correlations that cannot be explained by classical means. For example, a bipartite pure state can be entangled if it cannot be expressed in the product form mentioned earlier. In mixed states, entanglement is indicated if the state cannot be written as a convex combination of the product states [10].

3.4.3. Importance in Quantum Information

Determining whether a state is separable or entangled is crucial in quantum information theory because entangled states are essential resources for tasks such as quantum teleportation, superdense coding, and quantum cryptography. The problem of determining separability is known to be NP-hard, which makes it computationally challenging.

- Separable States: Can expressed as a combination of product states that exhibit classical correlations.
- Non-separable (entangled) States: Can not be decomposed into product states; exhibit quantum correlations that defy classical explanations.

Understanding these distinctions is vital for exploring the implications of quantum mechanics for various applications and theoretical frameworks.

3.5. Applications of Entanglement

Entanglement, a fundamental phenomenon in quantum mechanics, has numerous applications in various fields, particularly in quantum computing and quantum communication [5]. Some key applications are as follows:

i. Quantum Computing

- Quantum Parallelism: Entanglement allows quantum computers to perform multiple calculations simultaneously. By entangling qubits, quantum computers can manipulate many qubits in a single operation, significantly enhancing computational power compared to classical computers [41].
- Quantum Algorithms: Entangled states are essential for implementing quantum algorithms that outperform their classical counterparts. For instance, Shor's algorithm for factoring large numbers and Grover's algorithm for searching unsorted databases leverage entanglement to achieve exponential speedup over classical algorithms.
- Quantum Teleportation: This process enables the transfer of quantum states between distant systems without moving the physical particles. By entangling qubits at a source location and measuring the state of the original qubit, information can be transmitted effectively to recreate the state at the destination [42].
- Quantum Error Correction: Entanglement is crucial for developing error-correcting codes to protect quantum information from decoherence and other errors. By entangling multiple physical qubits, it is possible to create logical qubits that can correct errors without losing information [43].

ii. Quantum Communication

- Quantum Cryptography: Entanglement underpins protocols such as Quantum Key Distribution (QKD), which ensures secure communication channels. The security of QKD relies on the principles of entanglement and the impossibility of eavesdropping without detection [44].

- Distributed Quantum Computing: In scenarios where multiple quantum computers collaborate, entangled states can facilitate communication and coordination between them, thereby enhancing computational efficiency and capabilities.

iii. Fundamental Physics Research Testing Quantum Mechanics: Experiments involving entangled particles help test the foundations of quantum mechanics, including Bell's theorem and non-locality. These experiments provide insights into the nature of reality and the limitations of classical physics Quantum Sensors [45].

iv. Enhanced Measurement Techniques: Entangled states improve the measurement precision in various sensing applications. Quantum sensors that utilize entangled particles can achieve a higher sensitivity than classical sensors, making them valuable in fields such as gravitational wave detection and magnetic field sensing [46].

The applications of entanglement extend far beyond theoretical interest; they are pivotal for advancing technology and understanding fundamental physics. As research continues, the potential for new applications in quantum technologies remains vast and promising transformative across multiple domains [47].

4. USING QUANTUM SIMULATORS TO INVESTIGATE QUANTUM ENTANGLEMENT

Quantum simulators can address a variety of significant scientific challenges and greatly influence technological advancements. They are particularly adept at modeling quantum materials which remain poorly understood despite extensive research efforts. Additionally, these simulators promise breakthroughs in accurately simulating chemical processes that are currently beyond the reach of the most sophisticated supercomputers. They also provide access to extreme conditions relevant to high-energy particle physics and cosmology, which are difficult or impossible to replicate in laboratory settings [48]. Furthermore, quantum simulators offer an unparalleled level of control over spatial and temporal parameters, along with enhanced measurement precision and detail. This allows researchers to gain deeper insights into complex quantum systems than previously possible [49]. Recently, Google, IBM, and Intel have introduced quantum computers with capacities of 72, 53, and 49 qubits, respectively. Nevertheless, these state-of-the-art quantum computers still do not possess sufficient qubits to adequately apply error correction codes. Additionally, the noise inherent in quantum systems poses significant challenges for the advancement of quantum computing [50].

4.1. The IBM Q Systems

In 2016, IBM introduced the IBM Quantum Experience cloud platform, providing the public with unprecedented access to quantum computers for the first time. Users can currently utilize eight devices, that feature one, five, or 15 qubits, to support their development efforts. All quantum computers created by IBM employ a constrained set of gates, including arbitrary single-qubit gates and two-qubit CNOT gates. This limited collection of gates enables the execution of any conceivable quantum computation. [51].

However, IBM Q systems, like many quantum computers based on superconducting qubits, face significant challenges owing to limited connectivity among qubits. This constraint can affect the efficiency and complexity of quantum algorithms implemented in these systems [21].

4.2. Qiskit Framework

Qiskit is an open-source framework designed to develop software quantum computing. One of its key components, Qiskit Aqua, allows developers to implement quantum algorithms, while another component, Qiskit Terra, provides a transpiler that handles essential tasks such as decomposing quantum gates, mapping logical qubits to their physical counterparts, and optimizing circuits [52]. The transpiler incorporates modular passes for transforming circuits and employs a pass manager to organize these passes and manage their interactions. Users can control the pass manager to apply specific optimizations to their circuits. Four predefined pass managers correspond to optimization levels from 0 to 3, with higher levels involving more comprehensive optimization processes, resulting in longer transportation times. All quantum programs operate on qubits using gates. A single qubit can be represented visually using a Bloch Sphere with a radius of 1 as shown in Fig.1. The state of the qubit is determined by the point of intersection between a vector, that extends from the center of the sphere to its surface. The apex of the sphere represents the state$|0\rangle$, while the bottom point reflects the state $|1\rangle$. Gates alter the qubit's location on the sphere's surface to perform computations [53] $|1\rangle$.

Key Features of Qiskit

- Modular Design: Qiskit's architecture allows for flexibility and modularity, enabling users to effectively customize their quantum circuits.
- Transpilation: The transpiler optimizes quantum circuits for specific hardware constraints, thereby ensuring compatibility with various quantum devices.
- Visualization: The Bloch Sphere representation provides an intuitive way to understand qubit states and their transformations.

- Optimization Levels: Users can choose from different optimization levels based on their needs, balancing circuit efficiency and transportation time.

Qiskit serves as a comprehensive framework for developing quantum algorithms and applications, providing essential tools for both beginners and experienced researchers in the field of quantum computing [54].

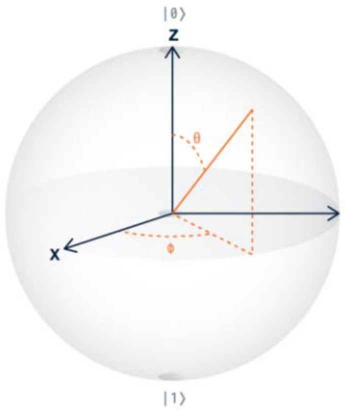


Figure 1. The Bloch Sphere visually represents a qubit, with the axes labeled as$X, Y, and Z.$ The state $|0\rangle$ is located at the top of the sphere, while $|1\rangle$ is at the bottom. The solid orange vector illustrates the position of the qubit on the sphere's surface. The angle $|\phi\rangle$ reflects a state of superposition, indicating that the qubit's value lies between $|0\rangle$ and $|1\rangle$. Meanwhile, the angle $|\phi\rangle$ signifies rotation around the Z axis, representing the qubit's phase.

4.2.1. IBM Quantum Composer

IBM Quantum Composer is a graphical user interface designed to compose, construct, and execute quantum circuits. Users can easily drag quantum gates onto qubits to build circuits. The interface provides real-time visualizations of quantum states, displaying probability distributions and representations on the Bloch sphere or state vector units as the circuit is constructed. This feature allows users to visually grasp the effects of applying various gates to qubits [55]. Additionally, as users create their circuits, Composer generates the corresponding Qiskit Python code, facilitating a standardized notation for quantum gates. Once the circuit is complete, users can execute it on actual IBM Quantum systems based on the selection of available machines. After execution, the measurement results can be reviewed on the Jobs webpage, accessible from the IBM Quantum front page [56].

Features

The user-friendly interface, characterized by its drag-and-drop functionality, significantly simplifies the circuit design, rendering it accessible to individuals with diverse expertise in quantum programming. Real-time visualization enables users to observe the immediate effects of circuit modifications on qubit states, thereby enhancing their comprehension of quantum mechanics principles. In addition, the automatic generation of the Qiskit code streamlines the transition from visual design to programming, thereby facilitating a seamless workflow. Users can execute their circuits on IBM's quantum computers, thereby gaining practical experience using authentic quantum systems. Finally, the analysis of measurement outcomes through the Jobs webpage allows for further experimentation and deepens the learning experience [57].

5. ENTANGLEMENT WITH IBM QUANTUM LAB AND QISKIT

In the process of creating entanglement, the fundamental quantum concept of entanglement. A key example is the quantum state.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{15}$$

This illustrates perfect correlations between two qubits, even when considering the experimental noise. This means that if qubit q_0 is measured and found to be in the state $|0\rangle$, it can be inferred that qubit q_1 is also in the state $|0\rangle$. Similarly, if q_0 is measured in the superposition state, then $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, it follows that q_1 will also be in the state $|+\rangle$. These correlations emphasize the non-classical characteristics of entangled states, wherein measuring one qubit instantaneously determines the state of its entangled partner, regardless of the distance between them.

5.1. Pure States

Pure states are those in which the quantum state can be precisely defined at every moment in time. For instance, if a single qubit $|q\rangle$ is initialized in the state $|0\rangle$ and a Hadamard gate is subsequently applied, the final state can be expressed as:

$$|q\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle \tag{16}$$

In this state, if a measurement is performed the outcome is probabilistic. Specifically, there is a 50% probability of measuring the state $|0\rangle$ and a 50% probability of measuring the state $|1\rangle$. This illustrates how pure states can exhibit superposition, leading to uncertainty in the measurement outcomes despite having a well-defined quantum state before measurement.

However, before performing any measurements, one can assert with 100% certainty that if the qubit initialization process and the Hadamard gate are ideal, the resulting quantum state will always be $|+\rangle$. This means that there is no uncertainty regarding the final state of the qubit; thus, we can confidently classify $|q\rangle$as a pure state.

Pure states are characterized by complete knowledge of their quantum state at all times, allowing for precise predictions of their behavior under ideal conditions. In general, a pure state comprising n-qubits can be articulated in the standard state vector notation as a linear combination of basis states. Specifically, the state can be represented as follows:

$$|\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \tag{17}$$

where $N = 2^n$. Another approach to represent this pure quantum state is through a matrix formulation. This can be achieved using the density operator representation, which is defined as follows:

$$\rho = |\psi\rangle\langle\psi|. \tag{18}$$

Here, the term $|\psi\rangle\langle\psi|$ represents the outer product of state ψ and itself.

$$\rho = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \begin{bmatrix} \alpha_0^* & \alpha_1^* & \dots & \alpha_N^* \end{bmatrix} \tag{19}$$

$$\rho = \begin{bmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* & \dots & \alpha_0\alpha_N^* \\ \alpha_1\alpha_0^* & |\alpha_1|^2 & \dots & \alpha_1\alpha_N^* \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_N\alpha_0^* & \alpha_N\alpha_1^* & \dots & |\alpha_N|^2 \end{bmatrix} \tag{20}$$

For instance, let us examine the following two-qubit pure state, which is maximally entangled.

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{21}$$

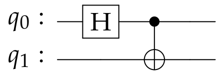
The density matrix representation for this state can then be expressed as:

$$\rho_{AB} = |\psi_{AB}\rangle \langle\psi_{AB}|$$

$$\rho_{AB} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \right) \tag{22}$$

$$\rho_{AB} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

In Qiskit, the quantum "info module" can utilized to represent quantum states in either the state vector notation or density matrix form. For ease of use, this module was imported into q_i as shown in ship 1.



*ship*1 : *Pure state*

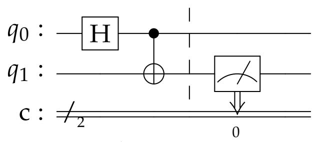
5.2. Mixed States

Mixed states are characterized by statistical ensembles of different quantum states, meaning that they represent a combination of various pure states rather than a single, well-defined state. Unlike pure states, which can be expressed as linear superpositions of normalized state vectors, mixed states reflect uncertainty regarding the exact state of the system.

Consider, once again, the two-qubit entangled state:

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|0_A 0_B\rangle + |1_A 1_B\rangle \right) \tag{23}$$

Here subscripts A and B are explicitly used to label the qubits associated with registers q_1 and q_0, respectively. Now, let us assume that immediately after preparing our state $|\psi_{AB}\rangle$ it performs a measurement on register q_1, as shown below in ship 2.



*ship*2 : *Mixed state*

5.3. Two-Qubit Correlated Observables

In quantum mechanics, an observable is represented by a Hermitian operator, which is a matrix with real eigenvalues that correspond to the possible outcomes of a measurement. The eigenvectors of this operator represent the states in which the system collapses upon the measurement. Formally, if we denote an observable by A, it can be expressed as.

$$A = \sum_j a_j \left| a_j \right\rangle \left\langle a_j \right| \tag{24}$$

where $|a_j\rangle$ is the eigenvector of the observable with result a_j. The expectation value of this observable is given by:

$$\langle A \rangle = \sum_j a_j \left| \langle \psi \mid a_j \rangle \right|^2 = \sum_j a_j \operatorname{Pr}\left(a_j \mid \psi \right) \tag{25}$$

There is a standard relationship between the average (expectation value) and probability.

For a two-qubit system, the following are important two-outcome (± 1) single-qubit observables:

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$X = |+\rangle\langle+| - |-\rangle\langle-|$$

These are also commonly referred to as Pauli Z and X operators respectively. These can be further extended to the two-qubit space as follows:

$$\langle I \otimes Z\rangle = \Pr(00|\psi) - \Pr(01|\psi) + \Pr(10|\psi) - \Pr(11|\psi)$$

$$\langle Z \otimes I\rangle = \Pr(00|\psi) + \Pr(01|\psi) - \Pr(10|\psi) - \Pr(11|\psi)$$

$$\langle Z \otimes Z\rangle = \Pr(00|\psi) - \Pr(01|\psi) - \Pr(10|\psi) + \Pr(11|\psi)$$

$$\langle I \otimes X\rangle = \Pr(++|\psi) - \Pr(+-|\psi) + \Pr(-+|\psi) - \Pr(--|\psi)$$

$$\langle X \otimes I\rangle = \Pr(++|\psi) + \Pr(+-|\psi) - \Pr(-+|\psi) - \Pr(--|\psi)$$

$$\langle X \otimes X\rangle = \Pr(++|\psi) - \Pr(+-|\psi) - \Pr(-+|\psi) + \Pr(--|\psi)$$

$$\langle Z \otimes X\rangle = \Pr(0+|\psi) - \Pr(0-|\psi) - \Pr(1+|\psi) + \Pr(1-|\psi)$$

$$\langle X \otimes Z\rangle = \Pr(+0|\psi) - \Pr(+1|\psi) - \Pr(-0|\psi) + \Pr(-1|\psi)$$

5.4. Bell State(s)

Two quantum systems are considered entangled when the values of the specific properties of one system are non-classically correlated with the corresponding values for another system. Bell states, which represent maximally entangled states of two qubits, are a key example of this phenomenon.

$$|\beta_{xy}\rangle = \frac{|0y\rangle + (-1)^x|1y^-\rangle}{\sqrt{2}} \tag{26}$$

The general form of the Bell state is as shown above. In this context,X and Y are referred to as phase and parity bits, respectively. One of the Bell states is x= 0, y = 1. For the specific case where $x = 0$ and $y = 1$, the Bell state simplifies to:

$$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}\left(|0,y\rangle + (-1)^x|1,1-y\rangle\right) \tag{27}$$

This represents a maximally entangled state of two qubits. In this state:

- If one qubit is measured and found to be in state $|0\rangle$, the other qubit will be in state$|1\rangle$
- Conversely, if the first qubit is measured and found to be in state $|1\rangle$, the second qubit will be in state $|0\rangle$

5.4.1. Properties of the Bell State

- Maximal Entanglement: State $|\beta_{01}\rangle$ is maximally entangled, meaning that the measurement outcomes of the two qubits are perfectly correlated.
- Measurement Outcomes: Each qubit has an equal probability (50%) of being measured in either state $|0\rangle$ or $|1\rangle$. However, the measurement of one qubit immediately determines the state of another.
- Quantum Communication: This entangled state can be utilized in various quantum communication protocols, such as quantum teleportation and superdense coding.

5.4.2. Creating Bell States

To create a Bell state-like $|\beta_{01}\rangle$, you typically start with two qubits in the state $|00\rangle$. A Hadamard gate can then be applied to one qubit followed by a CNOT gate:
Hadamard Gate: Apply a Hadamard gate to the first qubit:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

CNOT Gate: Use the first qubit as control and the second as a target for a CNOT gate:

$$\mathrm{CNOT}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle\right) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

To achieve $|\beta_{01}\rangle$, it is necessary to apply an additional operation (such as an X gate) on the second qubit after creating the initial Bell pair.

5.4.3. How to Create Bell State Using IBM Quantum Circuits

After reviewing the background and the necessary mathematics, simple quantum circuits can be constructed to create Bell states. As discussed previously, the states $(|0\rangle, |1\rangle)$and$(|+\rangle, |-\rangle)$ form an orthonormal basis for the two-dimensional complex vector spaceC^2. Another orthonormal basis set can be formed using the states$(|+\rangle, |+\rangle)$and$(|-\rangle, |-\rangle)$, defined as follows:

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad ,|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

The Hadamard operator is a crucial component in quantum computing, particularly for creating superposition and entangled states such as Bell states. The Hadamard matrix, often referred to as the Hadamard gate, transforms the standard basis states ($|0\rangle$ and $|1\rangle$) into the superposition states ($|+\rangle$ and $|-\rangle$).

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \quad ;$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

Following the transformation described above, it can be stated that the Hadamard operator will alter the state of a qubit as follows: $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to as below:

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$H|\Psi\rangle = \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

In general, the Hadamard operator can be employed to generate a superposed state. Let us now outline the steps for creating an entangled state. Bell states that two qubits are involved, and from the previous discussion, it is known that one of the simplest two-qubit gates is the CNOT or CX gate. The CX gate operator has been previously explored in detail, and its outer product representation is as follows:

$$CX = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

Let us consider a thought experiment in which the control qubit is in a superposed state. Let's say $|C\rangle = |0\rangle + |1\rangle$ and the target qubit is in $|0\rangle$ state.

$$CX(|00\rangle + |10\rangle) = (|00\rangle\langle 00| + |01\rangle\langle 01|$$

$$+|10\rangle\langle 11| + |11\rangle\langle 10|)(|00\rangle + |10\rangle)$$

$$CX(|00\rangle + |10\rangle) = |00\rangle + |11\rangle = \sqrt{2}\beta_{00}$$

So if it applies the CNOT gate to this system $(|00\rangle + |10\rangle)$, it will get — $|00\rangle + |11\rangle$. Returning to the general expression for Bell states in the previous discussion, it is evident that this represents one of the Bell states multiplied by a factor. Therefore, entangled qubits can created if it begin with a Hadamard gate to generate the superposition state and use this state as a control bit for the CNOT gate as show in ship 3. It now verifies this by writing as code.

Qiskit Code to Create a Bell State

```
import qiskit
#state vector of required qubits
is generated
from qiskit.quantum_info import
Statevector

# plot the qubits in a qsphere
from qiskit.visualization
import plot_state_qsphere

# show the input state 00
sv = Statevector.from_label('00')

plot_state_qsphere(sv.data)
#plot the previous state


# generate the circuit's bell state
mycircuit = qiskit.QuantumCircuit(2,2)
mycircuit.h(0)
mycircuit.cx(0,1)
mycircuit.draw('mpl')
```

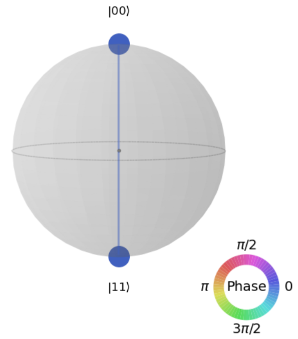


Figure 2. The bell state in the Bloch Sphere

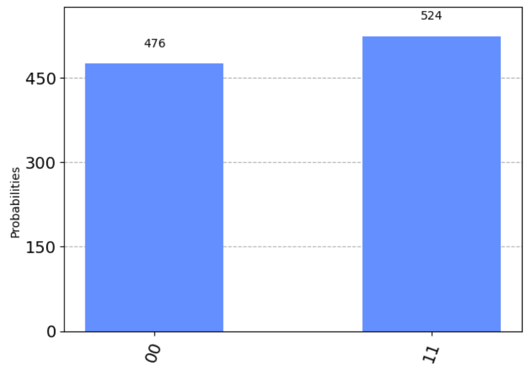
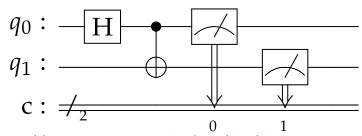


Figure 3. Measurement of bell circuit



ship3 : Bell state circuit by kiskit

```
# show the output vector state
new_sv = sv.evolve(mycircuit)
print(new_sv)
# show the plot of  the output qubits
plot_state_qsphere(new_sv.data)

Statevector([0.70710678+0.j,  0.
+0.j,  0.           +0.j,
          0.70710678+0.j],
        dims=(2, 2))
```

Both qubits are measured. The results will show correlations indicative of entanglement.

A histogram of measurement outcomes is plotted, which should show roughly equal probabilities for measuring |00⟩ and |11⟩.

```
# simulate  with 1000 iterations
counts = new_sv.sample_counts(shots=1000)

from qiskit.visualization import plot_histogram
plot_histogram(counts) # plot the output istogram
```

5.4.4. CHSH Inequality

In the context of the CHSH inequality, we measure the correlators of four observables: $A'$and A on qubit q_0, and

B and B' on q_1, on qubit , each having eigenvalues of ± 1. The CHSH inequality states that no local hidden variable theory can satisfy the following condition:

$$|C| > 2$$

where

$$C = \langle B \otimes A \rangle + \langle B \otimes A' \rangle + \langle B' \otimes A' \rangle - \langle B' \otimes A \rangle.$$

What would this look like with some hidden variable model under the locality and realism assumptions from above? C then becomes

$$C = \sum_{\lambda} P(\lambda) \{ B(\lambda)[A(\lambda) + A'(\lambda)] + B'(\lambda)[A'(\lambda) - A(\lambda)]$$

and $[A(\lambda) + A'(\lambda)] = 2$ (or 0) while $[A'(\lambda) - A(\lambda)] = 0$ (or 2) respectively. That is, $|C| = 2$, and noise will only make this smaller. If it measures a number greater than 2, the above assumptions cannot be valid. (This is a perfect example of one of those astonishing counterintuitive ideas one must accept in the quantum world.) For simplicity, it chooses these observables to be

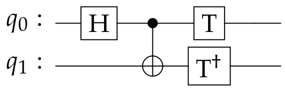
$$C = \langle Z \otimes Z \rangle + \langle Z \otimes X \rangle + \langle X \otimes X \rangle - \langle X \otimes Z \rangle.$$

Z is measured in the computational basis, and X in the superposition basis (H is applied before measurement). The input state

$$|\psi(\theta)\rangle = I \otimes Y(\theta) \frac{|00\rangle + |11\rangle}{\sqrt(2)} =$$

$$\frac{\cos(\theta/2)|00\rangle + \cos(\theta/2)|11\rangle + \sin(\theta/2)|01\rangle - \sin(\theta/2)|10\rangle}{\sqrt{2}}$$

is swept vs. θ (think of this as allowing us to prepare a set of states varying in the angle θ). Note that the following demonstration of CHSH is not loophole-free.



ship4 : CHSH state

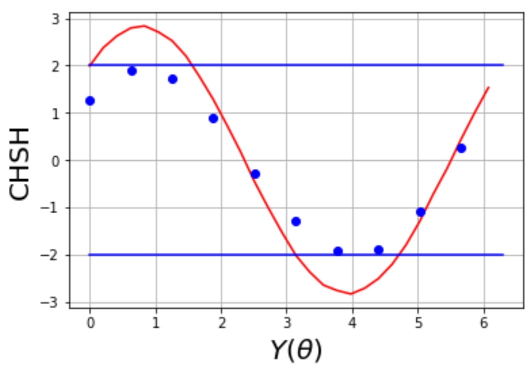


Figure 4. CHSH Inequality

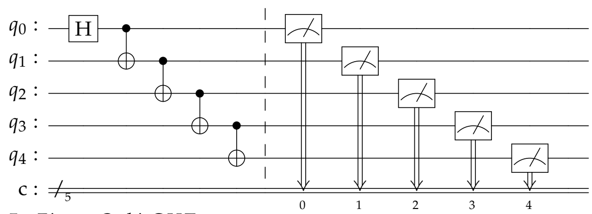
Here is the CHSH data:

$$4[1.257812, 1.90039062, 1.73632812, 0.9023437, \\ -0.27148437, -1.2929687, -1.91601562, \\ -1.8789062, -1.0742187, 0.26757812]$$

Despite the presence of loopholes in its demonstration, it can seem that this experiment is compatible with quantum mechanics as a theory with no local hidden variables.

5.4.5. Five-Qubit GHZ States

What does entanglement look like beyond two qubits? An important set of maximally entangled states are known as GHZ states (named after Greenberger, Horne, and Zeilinger). These are the states of the form. $|\psi\rangle = (|0...0\rangle + |1...1\rangle) / \sqrt{2}$. The Bell state previously described is merely a two-qubit version of a GHZ state. The next cells prepare GHZ states of two, three, and four qubits.



ship5 : Five − QubitGHZstate

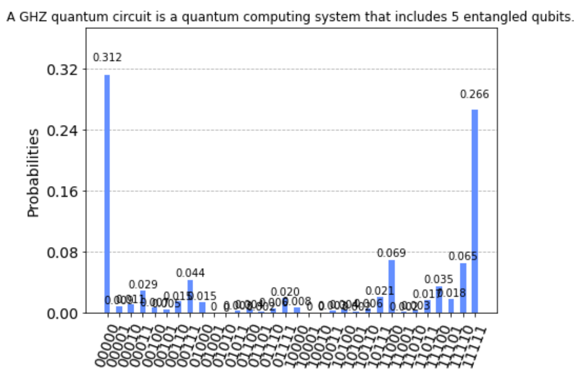


Figure 5. GHZ state

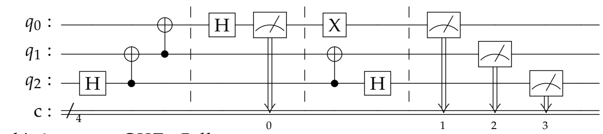
5.5. By IBM convert GHZ states to Bell states

This scenario utilizes mid-circuit measurement to illustrate the transformation of GHZ states into Bell states. It also shows how these measurements can impact the behavior of the quantum state.

Consider the standard three-qubit GHZ state represented as: :$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. If one measures one of the three qubits in the computational (z) basis without any strategy, the remaining two qubits end up in an unentangled state. However, it is important to recognize that a GHZ state can be expressed as (excluding normalization constants):

$$|000\rangle + |111\rangle = [|00\rangle + |11\rangle] \otimes (|0\rangle + |1\rangle) + [|00\rangle - |11\rangle] \otimes (|0\rangle + |1\rangle) = [|00\rangle - |11\rangle] \otimes |+x\rangle + [|00\rangle - |11\rangle] \otimes |-z\rangle,$$

It may appear that if any one of the three qubits is measured in the x-basis, the states of the two remaining qubits correspond to one of two possible Bell states, indicating that these qubits remain maximally entangled. Consequently, interference can be employed to convert these Bell states back into the computational basis. The circuit below achieves this by mapping $[|00\rangle + |11\rangle] \rightarrow |00\rangle$ and $[|00\rangle - |11\rangle] \rightarrow |10\rangle$. To demonstrate that the measurement occurs mid-circuit, the flag qubit—which records the outcome of the x-basis measurement will be flipped and then re-measured. The results will be stored in different classical bits to facilitate the simultaneous display of both outcomes.



$ship6 : convertGHZtoBellstates$

First, simulate to verify that it built the circuit correctly.
$Simmeasure1result : \{'1' : 525, '0' : 499\}$

$Simmeasure2result : \{'100' : 525, '001' : 499\}$

Now it tries on real hardware:
$Meas1result : \{'0' : 2050, '1' : 1950\}$
$Meas2result : \{'000' : 1320, '001' : 954, '010' : 140, '011' : 187, '100' : 268, '101' : 925, '110' : 103, '111' : 103\}$

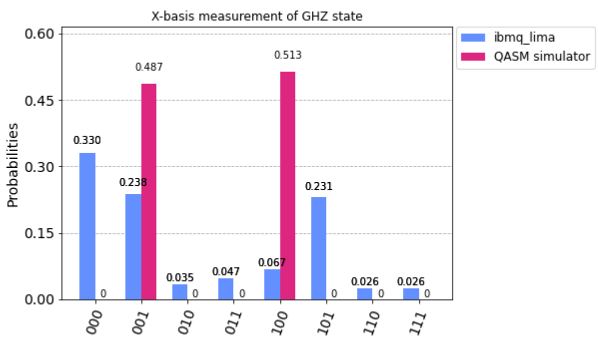


Figure 6. X-basis measurement of GHZ state

Now, let's explore the outcome when the initial measurement is conducted on a computational basis instead of an x basis, destroying the entanglement. Since the state following this measurement is no longer a Bell pair, it would anticipate that our transformation back to the computational basis will produce a different result.

$Simmeasresult : \{'101' : 270, '100' : 246, '000' : 245, '001' : 263\}$
$Measresult : \{'000' : 832, '001' : 828, '010' : 224, '011' : 150, '100' : 732, '101' : 819, '110' : 248, '111' : 167\}$

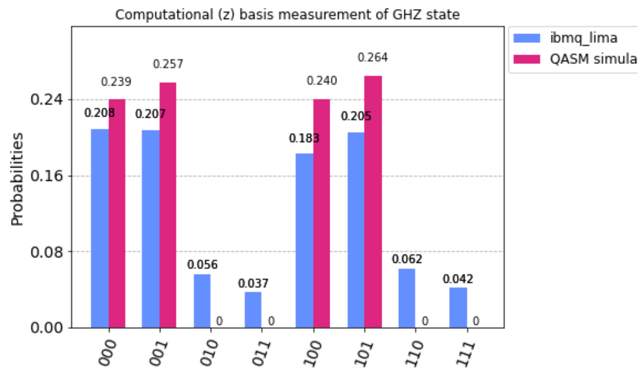


Figure 7. Computational (z) basis measurement of GHZ state

6. CONCLUSION

Entanglement is one of the most intriguing phenomena in quantum physics and forms the backbone of quantum information theory, quantum computation, and quantum communication. This study explores various topics related to both the theory and practice of entanglement, emphasizing the capabilities of IBM Quantum as a platform for understanding and implementing quantum computing and algorithms. IBM Quantum provides a reliable entry point for many individuals seeking to engage in quantum technologies. This platform demonstrated that publicly available quantum entanglement can effectively run smaller-scale entangled states with good accuracy. For instance, experiments have successfully generated and verified entangled states, including GHZ states, on various IBM processors, showcasing their ability to create complex entangled systems. However, hardware limitations persist across all quantum computers. Constructing and implementing quantum entanglement in a quantum circuit on a real quantum machine is not as straightforward as it may seem. Factors such as gate fidelity, error rates, and decoherence pose significant challenges. Despite these obstacles, ongoing advancements in quantum technology and error mitigation strategies continue to enhance the performance of quantum systems. In summary, while IBM Quantum has made significant strides in enabling access to quantum computing and demonstrating entanglement, challenges remain that require continued research and innovation. The exploration of entanglement not only deepens our understanding of quantum mechanics but also paves the way for practical applications that could revolutionize various fields through enhanced computational capabilities.

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