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Rigorous Spectral Analysis of a Schrödinger Operator with Fractional and Logarithmic Perturbations: Fixed-Point Formulation and Explicit Constants

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Abstract

We present a complete spectral analysis of the Schrödinger operator

$$\mathcal{O}_0 = -\frac{d^2}{dx^2} + \frac{\mu(\mu+1)}{x^2} + \omega^2 x^2$$

on the half-line $L^2(0,\infty)$, perturbed by two unbounded terms: a fractional power $\mathcal{O}_0^\sigma$ with $0 < \sigma < 1$, and a logarithmic term $\gamma \log x$. We first prove the complete spectral analysis of $\mathcal{O}_0$ via reduction to Laguerre's equation, and establish that the minimal, maximal, and Friedrichs operators coincide (the limit-point property at both endpoints). We then prove relative boundedness of both perturbations with relative bound zero, and apply the Kato–Rellich theorem to obtain self-adjointness of the full operator, with purely discrete spectrum.

We provide an explicit closed-form formula for the logarithmic matrix elements $V_{mn}^{\log} = \langle \psi_m, \log x\, \psi_n \rangle$ in terms of the digamma function, prove a uniform bound $|V_{n,n+r}^{\log}| \le 1/(2|r|)$, and derive the complete second-order expansion of the eigenvalues, which includes both a logarithmic term $-\alpha(\log n + \gamma_E - \log 2)/(16\omega H_n^*(n+1)^2)$ with $\alpha\mu + 1/2$ and an algebraic term $-\lambda\pi^2\sigma(1-\sigma)(4\omega n)^{\sigma-2}/(24H_n^2)$.

Next we develop a fixed-point formulation on the weighted Banach space Z_2', where we prove contraction of the operator $B_n(F)$ with constant 0.032, and obtain an exact implicit equation

$$E_n = e_n + \gamma V_{nn}^{\log} + F_n, \quad F_n = \Phi_n(F_n),$$

with $\mathrm{Lip}(\Phi_n) \le 2.95 \times 10^{-6}$ and $|F_n| \le 1.26 \times 10^{-5}\omega$. We further establish: (i) analyticity of $E_n(\lambda,\gamma)$ in a neighborhood of $(0,0)$ with radius $\rho = \min(\Lambda_{\text{gap}}/2, \omega/175)$; (ii) a rigorous second-order expansion with both algebraic and logarithmic corrections; (iii) generalization of the composite gap to all $\sigma \in (0,1)$.

All constants are explicit and numerically usable. The case $n = 0$ is treated separately due to the index constraint $s \ge 1$.

Article Info

Keywords:

Schrödinger operator , discrete spectrum , fractional perturbation , logarithmic perturbation , Kato–Rellich theorem , fixed-point formulation , explicit constants

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1. INTRODUCTION

1.1. Context and Motivation

Spectral perturbation theory for unbounded operators on the half-line is a cornerstone of functional analysis and mathematical physics. Since the foundational works of Kato [1] and Reed–Simon [2–4], powerful tools have been developed to study spectral stability [5, 6]. Nevertheless, two classes of perturbations still demand a finer analysis than the general framework provides. **Fractional perturbations.** Operators of the form $\mathcal{O}_0^\sigma$ with $0 < \sigma < 1$ arise naturally in anomalous diffusion [7], fractional quantum systems [8, 9], fractional Schrödinger equations with logarithmic nonlinearity [10–12], and random scattering [13]. They model non-local effects whose spectral signature is not captured by classical L^2 perturbation theory. Recent advances include the analysis of bound states [8], heat kernels [9], and the existence of normalized solutions [12].

Logarithmic perturbations. Terms of the form $\gamma \log x$ appear in the modified Calogero–Sutherland models [14–17], in Kolmogorov–Feller equations [18, 19], and in the de Sitter limit of quantum field theories [20]. They are unbounded at both endpoints of the half-line and therefore fall outside the scope of bounded-perturbation arguments.

Fix $\mu > 1/2$ and $\omega > 0$. The base operator is

$$\mathcal{O}_0 - \frac{d^2}{dx^2} + \frac{\mu(\mu+1)}{x^2} + \omega^2 x^2, \qquad x > 0, \tag{1}$$

with maximal domain

$$\mathcal{D}(\mathcal{O}_0) \left\{ f \in H^2(0,\infty) : \frac{f}{x^2} \in L^2(0,\infty), \right. \\ \left. x^2 f \in L^2(0,\infty) \right\}. \tag{2}$$

This operator combines a kinetic term, a singular Coulomb potential $\mu(\mu+1)/x^2$ (requiring $\mu > 1/2$ for uniqueness of the self-adjoint extension; see [21–23]), and a harmonic confinement $\omega^2 x^2$ producing purely discrete spectrum.

The full operator studied in this paper is

$$\mathcal{H}(\lambda,\gamma)\mathcal{O}_0 + \lambda \mathcal{O}_0^\sigma + \gamma \log x, \qquad 0 < \sigma < 1, \tag{3}$$

where $\mathcal{O}_0^\sigma$ is the fractional power defined via the spectral calculus $\mathcal{O}_0^\sigma \int_{\sigma(\mathcal{O}_0)} t^\sigma \, dE_t$, and the domain is $\mathcal{D}(\mathcal{H})\mathcal{D}(\mathcal{O}_0)$.

We impose three hypotheses on the coupling constants (λ,γ):

- **(H1)** *Monotonicity.* $\lambda > -\Lambda_{\text{mono}} - [\omega(2\mu+3)]^{1-\sigma}/\sigma$, ensuring strict monotonicity of the effective energies.
- **(H2)** *Uniform gap and positivity.* $|\lambda| \leq \Lambda_{\text{gap}} \min(\Lambda_{\text{gap}}^{\text{self}}, \Lambda_{G>0})$, where
$$\Lambda_{\text{gap}}^{\text{self}} \frac{[\omega(2\mu+3)]^{1-\sigma}}{4\sigma}, \qquad \Lambda_{G>0} \frac{1}{2\sigma(4\omega)^{\sigma-1}}.$$
- **(H3)** *Smallness of the logarithmic coupling.* $|\gamma| \leq \gamma_0^{\text{sharp}} \omega/175$.

These are conservative (safety margins 1.2–1.5 relative to sharp values), calibrated to ensure strict monotonicity of the effective energies, the composite spectral gap, and contraction of the fixed-point operator introduced below.

1.2. Relation to Previous Work

The base operator with a single logarithmic perturbation $\beta \ln x$ was studied in [24] using WKB asymptotics with the Langer correction [25, 26]. That work established self-adjointness, purely discrete spectrum, a sharp asymptotic expansion, a modified Weyl law, and the associated Green's function.

The present paper extends [24] in three directions:

(i) We add a fractional perturbation $\lambda \mathcal{O}_0^\sigma$ with $0 < \sigma < 1$;

(ii) We replace the WKB-based analysis by a rigorous fixed-point formulation based on Kato–Rellich theory [1, 27, 28], the Schur test [29], Stein interpolation [30, 31], and standard self-adjoint operator theory [2–6];

(iii) We establish analyticity of the eigenvalues in (λ,γ), a complete second-order expansion, and generalization of the composite gap to all $\sigma \in (0,1)$.

1.3. Main Results

The paper establishes the following results, all with explicit constants.

(1) Spectral analysis of $\mathcal{O}_0$. The operator $\mathcal{O}_0$ is self-adjoint on $\mathcal{D}(\mathcal{O}_0)$, has purely discrete spectrum, and its orthonormal eigenfunctions and eigenvalues are

$$\psi_n(x) = \mathcal{N}_n x^{\alpha+1/2} e^{-\omega x^2/2} L_n^{(\alpha)}(\omega x^2), \\ A_n = 4\omega n + \omega(2\mu+3), \quad n \geq 0, \tag{4}$$

where $\alpha\mu + 1/2$ and $\mathcal{N}_n \sqrt{2\omega^{\alpha+1} n!/\Gamma(n+\alpha+1)}$; the polynomials $L_n^{(\alpha)}$ are generalized Laguerre polynomials [32, 33]. The limit-point property of both endpoints (in the sense of Sturm–Liouville theory [21–23]) forces

$$\mathcal{O}_0^{\min} = \mathcal{O}_0^{\max} = \mathcal{O}_0^F = \mathcal{O}_0, \qquad \sigma_{\text{ess}}(\mathcal{O}_0) = \emptyset, \tag{5}$$

where $\mathcal{O}_0^F$ is the Friedrichs extension [34], and the Weyl criterion [35] gives discreteness. We also use Hardy's inequality [36] and standard operator-theoretic results [5, 6].

(2) Relative boundedness. For every $\epsilon > 0$,

$$\|\mathcal{O}_0^\sigma f\| \leq \epsilon \|\mathcal{O}_0 f\| + C_\epsilon \|f\|, \quad \|\log x \cdot f\| \\ \leq \epsilon \|\mathcal{O}_0 f\| + C'_\epsilon \|f\|, \tag{6}$$

for all $f \in \mathcal{D}(\mathcal{O}_0)$, with explicit constants

$$C_\epsilon^2 = \max\left(0, \sup_{t \geq A_0} (t^{2\sigma} - \epsilon^2 t^2)\right), \\ C'_\epsilon = \sqrt{L_{\delta,M}^2 + \epsilon^2/4}.$$

Both perturbations therefore have relative bound zero, and the Kato–Rellich theorem [1, 27, 28] yields self-adjointness, purely discrete spectrum, and boundedness from below for $\mathcal{H}(\lambda,\gamma)$.

(3) Logarithmic matrix elements. The elements $V_{mn}^{\log} \langle \psi_m, \log x \, \psi_n \rangle$ admit the closed form

$$V_{nn}^{\log} = \frac{1}{2}[\Psi(n+\alpha+1) - \log \omega], \tag{7}$$

$$V_{mn}^{\log} = -\frac{1}{2|m-n|}\sqrt{\frac{\max(m,n)!\,\Gamma(\min(m,n)+\alpha+1)}{\min(m,n)!\,\Gamma(\max(m,n)+\alpha+1)}},\quad m \neq n, \tag{8}$$

where $\Psi = \Gamma'/\Gamma$ is the digamma function [33, 37]. Moreover,

$$|V_{n,n+r}^{\log}| \leq \frac{1}{2|r|}, \qquad \forall r \neq 0,\ n+r \geq 0. \tag{9}$$

(4) Complete second-order expansion. The n-th eigenvalue admits the expansion

$$E_n = e_n + \gamma V_{nn}^{\log} + \gamma^2 E_n^{(2)} + O(\gamma^3), \tag{10}$$

where $e_n = A_n + \lambda A_n^{\sigma}$, and

$$\boxed{E_n^{(2)} = -\frac{\lambda\pi^2\sigma(1-\sigma)}{24H_n^2}(4\omega n)^{\sigma-2} - \frac{\alpha(\log n + \gamma_E - \log 2)}{16\omega H_n^*(n+1)^2} + O(n^{2\sigma-3}) + O(n^{-2}).} \tag{11}$$

Here $H_n 1 + \lambda\sigma(4\omega n)^{\sigma-1}$ and $H_n^* 1 + \lambda\sigma A_n^{\sigma-1} = H_n + O(1/n)$. The first term (algebraic) vanishes when $\lambda = 0$; the second (logarithmic) is present for all λ and is proportional to α.

(5) Analyticity. The function $(\lambda,\gamma) \mapsto E_n(\lambda,\gamma)$ is analytic in a neighborhood of $(0,0)$ with radius

$$\rho = \min\left(\frac{\Lambda_{\text{gap}}}{2}, \frac{\omega}{175}\right). \tag{12}$$

(6) Generalization to $\sigma \in (0,1)$. The composite gap

$$|D_{n,r}(F)| \geq 2\omega|r| + \frac{\omega}{2} \tag{13}$$

holds uniformly in $\sigma \in (0,1)$, where Λ_{gap} depends on σ.

(7) Fixed-point formulation. Introducing the weighted Banach space

$$Z_2' \left\{ u = (u_r)_{r\neq 0} : \|u\|_{Z_2'} \sup_{r\neq 0} \frac{|r|^2|u_r|}{(\log(2+|r|))^2} < \infty \right\}, \tag{14}$$

the eigenvalue problem for $\mathcal{H}(\lambda,\gamma)$ reduces to a contraction equation $u = A_n(F) + B_n(F)u$ on Z_2', with

$$\|B_n(F)\|_{Z_2' \to Z_2'} \leq 0.032. \tag{15}$$

Consequently, the n-th eigenvalue satisfies the exact implicit equation

$$\boxed{E_n = g_n + F_n = e_n + \gamma V_{nn}^{\log} + F_n, \quad F_n = \Phi_n(F_n),} \tag{16}$$

where $g_n = e_n + \gamma V_{nn}^{\log}$, and

$$\operatorname{Lip}(\Phi_n) \leq 2.95 \times 10^{-6}, \quad |F_n| \leq 1.26 \times 10^{-5}\omega. \tag{17}$$

The case $n = 0$ is treated separately because the index constraint $s \geq 1$ makes all sums one-sided, but the same constants apply.

(8) Summary of main constants. Table 1 collects the principal constants.

Table[1]: Main explicit constants. All constants are *rigorous bounds* unless stated otherwise. The safety factors are conservative relative to sharp values.

Constant	Value / Bound	Location
C_V (digamma difference)	0.8607 (sharp)	Lemma 4.1
C_1 (Schur row)	$5/2 = 2.5$ (exact)	Theorem 4.18
C_2 (Schur column)	$6697/1800 \approx 3.7206$ (exact)	Theorem 4.18
$\|K\|$ (Schur norm)	$\leq \sqrt{33485}/60 \approx 3.0498$	Theorem 4.18
C_2^{tail} (tail constant)	≤ 2 (rigorous)	Lemma 4.11
C_2^B (extended convolution)	≤ 22.4 (rigorous; sharp ≈ 22.2)	Lemma 4.16
$\|B_n(F)\|$ (contraction)	≤ 0.032	Proposition 4.30
C_{dec} (decay)	$\leq 0.429\|\gamma\|/c_\omega$	Theorem 4.32
L_u (Lipschitz)	$\leq 5.74 \times 10^{-4}/c_\omega$	Lemma 4.33
$\|\Phi_n\|_\infty$ (consistency)	$\leq 0.772\|\gamma\|^2/c_\omega$	Lemma 4.35
$\operatorname{Lip}(\Phi_n)$	$\leq 2.95 \times 10^{-6}$	Lemma 4.36
$\|F_n\|$ (correction size)	$\leq 1.26 \times 10^{-5}\omega$	Theorem 4.39
$\|A_0(F)\|$ ($n=0$ source)	$\leq 0.251\|\gamma\|/c_\omega$	Lemma 4.47

1.4. Novelty and Structure of the Paper

To the best of our knowledge, this is the first time that the following are presented simultaneously for an operator combining fractional and logarithmic perturbations with a singular Coulomb potential:

(1) An explicit closed-form formula for the logarithmic matrix elements $V_{mn}^{\log}$;
(2) A complete second-order expansion with both algebraic and logarithmic corrections;
(3) A rigorous proof of analyticity of $E_n(\lambda,\gamma)$ with radius $\rho = \min(\Lambda_{\text{gap}}/2, \omega/175)$;
(4) An exact fixed-point formulation with a uniform contraction constant 0.032;
(5) Explicit numerically usable constants;
(6) Generalization of the composite gap to all $\sigma \in (0,1)$.

The results are consistent with recent developments in fractional operators [38] and extend the analysis of [8, 11, 12]. A detailed comparison with the modern literature [39–49] is given in Section 5.

Structure of the paper. Section 2 establishes the spectral setup of $\mathcal{O}_0$: full spectral analysis via Laguerre's equation, closedness of the quadratic form, the limit-point property, and the equality of minimal, maximal, Friedrichs, and original operators. Section 3 proves relative boundedness of both perturbations, derives the closed-form matrix elements together with their uniform bound, and establishes the complete second-order expansion. Section 4 develops the convolution operator, the Schur test, Stein interpolation, the Extended Convolution Lemma, and the fixed-point formulation, with a separate treatment of the case $n = 0$. Section 5 discusses the results, compares with the literature, and lists open questions. Appendix A collects the constants and supplementary tables.

1.5. Notation

Throughout, $\log$ denotes the natural logarithm; $\Psi = \Gamma'/\Gamma$ is the digamma function; γ_E is the Euler–Mascheroni constant; and C denotes a generic positive constant whose value may change from line to line. The operators $\mathcal{O}_0$, $\mathcal{O}_0^{\sigma}$, and $\mathcal{H}$ are defined in Eq. (1), Definition 2.6, and Eq. (3); the eigenfunctions and eigenvalues of $\mathcal{O}_0$ are ψ_n and A_n; the unperturbed, effective, and exact energies are e_n, g_n, and E_n; and $\sigma(\cdot)$, $\sigma_{\text{ess}}(\cdot)$ denote the spectrum and essential spectrum. We use the working constants $c_\omega 2\omega$ and $R\omega/2$.

2. SPECTRAL SETUP

Conventions for this section.

- *Self-containedness.* Every result used in this section is either proven here completely, or cited from standard classical literature.
- *Constants.* All explicit constants are rigorous bounds unless stated otherwise. The classification is as follows:
  - *Exact/sharp values:* $C_V = 0.8607$ (sharp formula), $C_1 = 5/2$, $C_2 = 6697/1800$.
  - *Rigorous bounds:* $\|K\| \le \sqrt{33485}/60 \approx 3.0498$, $C_2^{\text{tail}} = 2$ (Lemma 4.11), $C_2^B \le 22.4$ (Lemma 4.16).
  - *Working bounds:* All constants derived from the fixed-point structure ($\|B_n(F)\|$, C_{dec}, L_u, $\|\Phi_n\|_\infty$, $\mathrm{Lip}(\Phi_n)$, $|F_n|$, $E_n^{(2)}$) inherit the conservatism of C_2^B.
- *Table notation.* In tables, the symbols $|\cdot|$, $\|\cdot\|$, and $[\cdot]$ are used instead of the raw symbols.
- *Convention on n_Δ.* Two definitions are used: $n_\Delta^{(2)} \lceil 2\alpha \rceil$ in Lemma 3.12, and $n_\Delta^{(4)} \max\left(\lceil 4\alpha \rceil, \lceil 3(1+\alpha) \rceil\right)$ in Lemma 3.15. When no ambiguity arises, we write $n \ge n_\Delta$.
- *General notation.* $\log$ denotes the natural logarithm; $\Psi\Gamma'/\Gamma$ the digamma function; γ_E the Euler–Mascheroni constant; $\alpha\mu + 1/2$ the shifted exponent; C a generic positive constant that may change from line to line. We use the working constants $c_\omega 2\omega$ and $R\omega/2$.

2.1. The Full Operator and Hypotheses

2.1.1. The Base Operator

Definition 2.1 (Base operator) *Let $\mu > 1/2$ and $\omega > 0$. On the Hilbert space $L^2(0,\infty)$ with inner product*

$$\langle f, g \rangle \int_0^\infty f(x)\overline{g(x)}\,dx, \tag{18}$$

define the base operator

$$\boxed{\mathcal{O}_0 - \frac{d^2}{dx^2} + \frac{\mu(\mu+1)}{x^2} + \omega^2 x^2, \qquad x > 0.} \tag{19}$$

Definition 2.2 (Operator domain) *The domain of $\mathcal{O}_0$ is*

$$\boxed{\begin{aligned} \mathcal{D}(\mathcal{O}_0) \Big\{ f \in H^2(0,\infty) : \frac{f}{x^2} \in L^2(0,\infty), \\ x^2 f \in L^2(0,\infty) \Big\}. \end{aligned}} \tag{20}$$

Remark 2.3 (Justification of the domain) *The set $\mathcal{D}(\mathcal{O}_0)$ in* Eq. (20) *is the correct maximal domain on which $\mathcal{O}_0 f \in L^2(0,\infty)$. Indeed, for $f \in \mathcal{D}(\mathcal{O}_0)$:*

(1) $-f'' \in L^2$ because $f \in H^2$;
(2) $\mu(\mu+1)f/x^2 \in L^2(0,\infty)$ if and only if $f/x^2 \in L^2(0,\infty)$, because $\mu(\mu+1)$ is a positive constant;
(3) $\omega^2 x^2 f \in L^2(0,\infty)$ if and only if $x^2 f \in L^2(0,\infty)$, because ω^2 is a positive constant.

Hence the sum of the three terms lies in L^2, so $\mathcal{O}_0 f \in L^2(0,\infty)$. The converse direction — that $\mathcal{D}(\mathcal{O}_0)$ contains all f for which the Sturm–Liouville expression $\mathcal{O}_0 f$ lies in L^2 (distributionally) — is established in Theorem 2.30 below.

Counterexample. *The weaker conditions $xf \in L^2$ and $f/x \in L^2$ do not suffice. Take $f_0(x)xe^{-x}$. We verify:*

- *$f_0 \in H^2(0,\infty)$: indeed $f_0'(x) = (1-x)e^{-x}$, $f_0''(x) = (x-2)e^{-x}$, and each lies in $L^2(0,\infty)$;*
- *$xf_0(x) = x^2 e^{-x}$, and $\int_0^\infty x^4 e^{-2x}\,dx < \infty$, so $xf_0 \in L^2$;*
- *$f_0(x)/x = e^{-x}$, and $\int_0^\infty e^{-2x}\,dx < \infty$, so $f_0/x \in L^2$.*

However $f_0(x)/x^2 = e^{-x}/x$, and

$$\int_0^1 \frac{e^{-2x}}{x^2}\,dx = \infty, \tag{21}$$

since $e^{-2x} \ge e^{-2}$ on $(0,1]$ and $\int_0^1 x^{-2}\,dx = \infty$. Hence $f_0/x^2 \notin L^2$ and $f_0 \notin \mathcal{D}(\mathcal{O}_0)$.

Remark 2.4 (Necessity of $\mu > 1/2$) *We show that $\mu > 1/2$ is necessary for the Frobenius branch compatible with the domain condition* Eq. (20). *The argument is a self-contained analysis of the equation $-f'' + \mu(\mu+1)f/x^2 + \omega^2 x^2 f = h$ near $x = 0^+$, and does not use any result from later in this section.*

Frobenius analysis at $x = 0^+$. *We look for solutions of the homogeneous equation $-f'' + \mu(\mu+1)f/x^2 + \omega^2 x^2 f = 0$ of the form $f(x) = Cx^s + O(x^{s+1})$ as $x \to 0^+$, with $C \ne 0$ and $s \in \mathbb{R}$. Substituting and retaining the leading-order terms (of order x^{s-2}), the lower-order terms $\omega^2 x^2 f = O(x^{s+2})$ are negligible, and we obtain the indicial equation*

$$-s(s-1) + \mu(\mu+1) = 0 \iff s^2 - s - \mu(\mu+1) = 0. \tag{22}$$

The discriminant is $1+4\mu(\mu+1) = (2\mu+1)^2$*, so the roots are*

$$s_{1,2} = \frac{1 \pm (2\mu+1)}{2}, \quad \text{so} \quad s_1 = \mu+1, \quad s_2 = -\mu. \quad (23)$$

L^2-local condition. *A function* x^s *lies in* $L^2(0,\epsilon)$ *if and only if*

$$\int_0^\epsilon x^{2s}\,dx < \infty \iff 2s > -1 \iff s > -1/2. \quad (24)$$

Since $\mu > 1/2$*, we have:*

- $s_1 = \mu+1 > 3/2 > -1/2$*: always in* $L^2(0,\epsilon)$*;*
- $s_2 = -\mu < -1/2$*: not in* $L^2(0,\epsilon)$*.*

Hence exactly one of the two linearly independent Frobenius solutions lies in $L^2(0,\epsilon)$*, namely the one associated with* $s_1 = \mu+1$*. The* L^2*-admissible branch has leading behaviour* $x^{\mu+1}$*.*

The domain condition. *The companion integral* $\int_0^\epsilon |x^{\mu-1}|^2\,dx = \int_0^\epsilon x^{2\mu-2}\,dx$ *converges if and only if* $2\mu-2 > -1 \iff \mu > 1/2$*. Therefore the domain condition* $f/x^2 \in L^2$ *is compatible with the leading behaviour of the* L^2*-admissible branch if and only if* $\mu > 1/2$*. This confirms that the threshold* $\mu > 1/2$ *in Definitions 2.1–2.2 is sharp for that purpose.*

The two borderline cases.
At $\mu = 1/2$. *Here* $\alpha = 1$*, so* $f/x^2 \sim Cx^{-1/2}$ *for the* L^2*-admissible branch and* $\int_0^\epsilon |f/x^2|^2\,dx \sim \int_0^\epsilon x^{-1}\,dx = \infty$*. Nevertheless, the endpoint* $x=0$ *remains of limit-point type (since* $2\mu = 1$ *and* $\int_0^\epsilon |x^{-\mu}|^2\,dx = \int_0^\epsilon x^{-1}\,dx = \infty$*), so no additional boundary condition is required to select a self-adjoint extension. The actual obstruction is that the domain condition* $f/x^2 \in L^2$ *excludes the eigenfunctions. The correct remedy is to relax the domain condition to a wider space (such as* $\mathcal{D}(\mathcal{O}_0^{\max})$*, which contains all* ψ_n *because* $\mathcal{O}_0\psi_n = A_n\psi_n \in L^2$*). See Remark 2.32 for details.*
At $\mu < 1/2$. *The endpoint* $x=0$ *is of limit-circle type (since* $2\mu < 1$ *and* $\int_0^\epsilon x^{-2\mu}\,dx < \infty$*) and an additional boundary condition is required to select a self-adjoint extension. This is essentially different from the case* $\mu = 1/2$*.*
Both borderline cases lie outside the scope of the present paper.

Remark 2.5 (Inclusion of $n=0$) *All main results are stated for all* $n \geq 0$*. Three exceptions apply at* $n = 0$*:*

(1) Ansatz and fixed-point equation. *Stated separately (Section 4.4) because the index constraint* $n+s \geq 0$ *forces* $s \geq 1$ *at* $n=0$*, making the sums one-sided.*
(2) Second-order correction formula. *The formula involving* H_n *requires* $n \geq 1$*, since* $H_n = 1 + \lambda\sigma(4\omega n)^{\sigma-1}$ *(see Definition 3.20) diverges at* $n=0$ *when* $\lambda \neq 0$ *(as* $\sigma - 1 < 0$ *implies* $(4\omega n)^{\sigma-1} \to \infty$*). When* $\lambda = 0$*, we have* $H_n = 1$ *for all* $n \geq 0$*, and no divergence occurs.*
(3) Definition of H_n. *Excluded at* $n=0$ *when* $\lambda \neq 0$ *(Remark 3.21).*

Note on constants. *Most constants are identical for* $n=0$ *and* $n \geq 1$*: the contraction constant* $\|B_n(F)\| \leq 0.032$*, the decay constant* $C_{\text{dec}} \leq 0.429|\gamma|/c_\omega$*, and the consistency bound* $\|\Phi_n\|_\infty \leq 0.772|\gamma|^2/c_\omega$*. Three constants differ:*

- *the Lipschitz constant* $L_u = 5.74 \times 10^{-4}/c_\omega$ *for* $n \geq 1$ *versus* $L_u^{(0)} = 6.7 \times 10^{-4}/c_\omega$ *for* $n=0$*;*
- *the derived Lipschitz constant* $\mathrm{Lip}(\Phi_n) \leq 2.95 \times 10^{-6}$ *for* $n \geq 1$ *versus* $\mathrm{Lip}(\Phi_0) \leq 3.45 \times 10^{-6}$ *for* $n=0$*;*
- *the source bound* $\|A_n(F)\| \leq 0.415|\gamma|/c_\omega$ *for* $n \geq 1$ *versus* $\|A_0(F)\| \leq 0.251|\gamma|/c_\omega$ *for* $n=0$*.*

See Lemma 4.33 and Remark 4.51 for the first two, and Lemma 4.47 for the third.

The three exceptions (1)–(3) listed at the beginning of this remark concern the form *of the equations, not the values of the main constants.*

2.1.2. Dimensional Analysis

Fix a reference length $x_0 > 0$ and interpret the logarithmic potential as $\log(x/x_0)$. Under the scaling $x \to x_0 u$:

$$\mathcal{O}_0 \xrightarrow{x=x_0u} \frac{1}{x_0^2}\left[-\frac{d^2}{du^2} + \frac{\mu(\mu+1)}{u^2} + \omega^2 x_0^4 u^2\right]. \quad (25)$$

Under the substitution, $\frac{d}{dx} = x_0^{-1}\frac{d}{du}$. Hence:

$$-\frac{d^2}{dx^2} \to -x_0^{-2}\frac{d^2}{du^2}; \quad \frac{\mu(\mu+1)}{x^2} \to x_0^{-2}\frac{\mu(\mu+1)}{u^2};$$
$$\omega^2 x^2 \to x_0^{-2}\omega^2 x_0^4 u^2.$$

This yields the stated formula.

Eigenvalues. $A_n(\omega) = x_0^{-2} A_n(\omega x_0^2)$. The spectrum changes with ω, but its structure (discreteness, orthogonality) is preserved.

Logarithmic potential. $\log(x/x_0) = \log x - \log x_0$. Adding $\gamma \log(x/x_0)$ to $\mathcal{H}$ is equivalent to adding $\gamma \log x$ and shifting all eigenvalues by $-\gamma \log x_0$.

Conclusion. Without loss of generality, we set $x_0 = 1$.

2.1.3. The Full Operator

Definition 2.6 (Fractional power) *For* $0 < \sigma < 1$*, the fractional power* $\mathcal{O}_0^\sigma$ *is defined by the spectral calculus*

$$\mathcal{O}_0^\sigma \int_{\sigma(\mathcal{O}_0)} t^\sigma\,dE_t, \quad (26)$$

where E_t *is the spectral measure of* $\mathcal{O}_0$*. The operator* $\mathcal{O}_0^\sigma$ *is positive, self-adjoint, commutes with* $\mathcal{O}_0$*, and has* *eigenvalues A_n^σ. Its domain is*

$$\boxed{\mathcal{D}(\mathcal{O}_0^\sigma)\left\{f \in L^2(0,\infty) : \int_{\sigma(\mathcal{O}_0)} t^{2\sigma}\, d\|E_t f\|^2 < \infty\right\}.} \tag{27}$$

The spectral-calculus definition does not require an explicit eigenbasis. After Theorem 2.13, it is equivalent to

$$\mathcal{D}(\mathcal{O}_0^\sigma) = \left\{ f = \sum_n c_n \psi_n \in L^2 : \sum_n A_n^{2\sigma}|c_n|^2 < \infty \right\}. \tag{28}$$

Since $A_n \geq A_0 > 0$ and $\sigma < 1$, we have $A_n^{2\sigma} \leq A_0^{2(\sigma-1)} A_n^2$, hence $\mathcal{D}(\mathcal{O}_0) \subseteq \mathcal{D}(\mathcal{O}_0^\sigma)$.

Definition 2.7 (Full operator) *The full operator studied in this paper is*

$$\boxed{\mathcal{H}(\lambda,\gamma)\mathcal{O}_0 + \lambda\mathcal{O}_0^\sigma + \gamma\log x, \qquad 0<\sigma<1,} \tag{29}$$

where $\lambda \in \mathbb{R}$ is the fractional coupling constant, $\gamma \in \mathbb{R}$ is the logarithmic coupling constant, and the domain is provisionally chosen as $\mathcal{D}(\mathcal{H})\mathcal{D}(\mathcal{O}_0)$. This is justified as follows: (i) Proposition 3.5 verifies that $\mathcal{H}$ is self-adjoint on this domain; (ii) Theorem 2.30 below shows that $\mathcal{D}(\mathcal{O}_0)$ is the unique self-adjoint domain for $\mathcal{O}_0$.

Domain of $\log x$. *Define*

$$\boxed{\mathcal{D}(\log x)\left\{f \in L^2(0,\infty) : \log x \cdot f \in L^2(0,\infty)\right\}.} \tag{30}$$

This is a self-adjoint domain because $\log x$ is a real, measurable function (multiplication operator). By Theorem 3.3, we have $\mathcal{D}(\mathcal{O}_0) \subseteq \mathcal{D}(\log x)$.

2.1.4. Hypotheses on the Coupling Constants

We impose three sets of conditions on (λ,γ).

(H1) Monotonicity. For strict monotonicity of the sequence $e_n A_n + \lambda A_n^\sigma$ in n:

$$\lambda > -\Lambda_{\text{mono}} - \frac{[\omega(2\mu+3)]^{1-\sigma}}{\sigma}. \tag{31}$$

(H2) Uniform gap and positivity. We introduce two thresholds:

$$\begin{aligned} &\text{(H2a) Uniform gap:} \quad |\lambda| \leq \Lambda_{\text{gap}}^{\text{self}} \frac{[\omega(2\mu+3)]^{1-\sigma}}{4\sigma}, \\ &\text{(H2b) Positivity of } H_n\text{:} \quad |\lambda| \leq \Lambda_{G>0} \frac{1}{2\sigma(4\omega)^{\sigma-1}}. \end{aligned} \tag{32}$$

The unified threshold is $\Lambda_{\text{gap}} \min(\Lambda_{\text{gap}}^{\text{self}}, \Lambda_{G>0})$.

(H3) Smallness of the logarithmic coupling. We introduce two thresholds:

$$\begin{aligned} &\text{(H3a) Basic contraction:} \quad |\gamma| \leq \gamma_0^{\text{FP}} \frac{2\omega}{175} \approx 0.0114\,\omega, \\ &\text{(H3b) Sharp contraction:} \quad |\gamma| \leq \gamma_0^{\text{sharp}} \frac{\gamma_0^{\text{FP}}}{2} = \frac{\omega}{175} \approx 0.00571\,\omega. \end{aligned} \tag{33}$$

Roles of the two thresholds. γ_0^{FP} is used in Proposition 3.5 (self-adjointness via Kato–Rellich), while γ_0^{sharp} is used in Sections 4.3–4.4 (fixed-point contraction). The factor $1/2$ is a safety margin for the fixed-point formulation.

Relation between (H1) and (H2a). Since $\Lambda_{\text{gap}}^{\text{self}} = \Lambda_{\text{mono}}/4$ and $\Lambda_{\text{gap}} \leq \Lambda_{\text{gap}}^{\text{self}}$,

$$\Lambda_{\text{gap}} \leq \frac{\Lambda_{\text{mono}}}{4} \leq \Lambda_{\text{mono}}. \tag{34}$$

Hence (H2) implies (H1) on both sides, and the effective range for λ is $-\Lambda_{\text{gap}} \leq \lambda \leq \Lambda_{\text{gap}}$.

Definition 2.8 (Global threshold Λ^*)

$$\boxed{\Lambda^* \inf_{\sigma\in(0,1)} \Lambda_{\text{gap}}^{\text{self}}(\sigma).} \tag{35}$$

This threshold is stated for completeness. Its explicit value, with $B\omega(2\mu+3)$, is

$$\Lambda^* = \begin{cases} 1/4, & B \geq e^{-1}, \\ -Be\log B/4, & 0 < B < e^{-1}. \end{cases} \tag{36}$$

The derivation is given at the end of this subsection.

Relation to Λ_{gap}. *Λ^* is a* global *threshold (independent of σ), while $\Lambda_{\text{gap}}(\sigma)$ depends on σ. The unified domain for applying all results is $|\lambda| \leq \min(\Lambda^*/2, \Lambda_{\text{gap}}/2)$; see Remark 2.12.*

Remark 2.9 (Intersection μ_c) *The two thresholds $\Lambda_{\text{gap}}^{\text{self}}$ and $\Lambda_{G>0}$ coincide when*

$$\frac{[\omega(2\mu+3)]^{1-\sigma}}{4\sigma} = \frac{1}{2\sigma(4\omega)^{\sigma-1}}. \tag{37}$$

Solution. *Multiply both sides by $4\sigma(4\omega)^{\sigma-1}$:*

$$[\omega(2\mu+3)]^{1-\sigma}(4\omega)^{\sigma-1} = 2. \tag{38}$$

Using $\omega^{1-\sigma}\omega^{\sigma-1} = 1$ and $4^{\sigma-1} = 2^{2(\sigma-1)}$:

$$(2\mu+3)^{1-\sigma}\cdot 2^{2(\sigma-1)} = 2 \implies (2\mu+3)^{1-\sigma} = 2^{3-2\sigma}. \tag{39}$$

Raising both sides to the power $1/(1-\sigma)$, and writing $(3-2\sigma)/(1-\sigma) = 2 + 1/(1-\sigma)$:

$$2\mu+3 = 2^2\cdot 2^{1/(1-\sigma)} = 4\cdot 2^{1/(1-\sigma)}. \tag{40}$$

Hence

$$\boxed{\mu_c \frac{4\cdot 2^{1/(1-\sigma)} - 3}{2}.} \tag{41}$$

Two cases.

- *For $\mu < \mu_c$: (H2a) dominates, i.e. $\Lambda_{\mathrm{gap}} = \Lambda_{\mathrm{gap}}^{\mathrm{self}}$.*
- *For $\mu > \mu_c$: (H2b) dominates, i.e. $\Lambda_{\mathrm{gap}} = \Lambda_{G>0}$.*

At $\sigma = 1/2$: $\mu_c = (4\cdot 4 - 3)/2 = 13/2 = 6.5$.

Remark 2.10 (Second γ threshold) *There is a second threshold*

$$\gamma_0^{\mathrm{gap}} \frac{\omega}{2C_V \log 3} \approx 0.5288\,\omega, \tag{42}$$

ensuring the composite gap (Lemma 4.24). Here $C_V = 0.8607$ is the explicit constant from Lemma 4.1.

Numerical check.

$$\frac{\gamma_0^{\mathrm{gap}}}{\omega} = \frac{1}{2C_V \log 3} = \frac{1}{2\cdot 0.8607\cdot 1.0986} = \frac{1}{1.8911} \approx 0.5288. \tag{43}$$

Since $\gamma_0^{\mathrm{sharp}}/\gamma_0^{\mathrm{gap}} = (\omega/175)/(0.5288\omega) \approx 1/92.5$, the controlling threshold is $\gamma_0^{\mathrm{sharp}}$.

Remark 2.11 (Failure of the global thresholds chain for $\mu > \mu_c$) *The thresholds Λ^* and $\Lambda_{\mathrm{gap}}(\sigma)\min(\Lambda_{\mathrm{gap}}^{\mathrm{self}}(\sigma), \Lambda_{G>0}(\sigma))$ are used in different contexts. One naturally expects the chain*

$$\frac{\Lambda^*}{2} \le \frac{\Lambda_{\mathrm{gap}}^{\mathrm{self}}(\sigma)}{2} \le \Lambda_{\mathrm{gap}}^{\mathrm{self}}(\sigma). \tag{44}$$

This chain is valid because $\Lambda^* \le \Lambda_{\mathrm{gap}}^{\mathrm{self}}(\sigma)$ for every $\sigma \in (0,1)$, by definition of Λ^* as an infimum. However, the chain does not extend to $\Lambda_{\mathrm{gap}}(\sigma)$ when $\mu > \mu_c$.

Explicit numerical example. *Take $\omega = 10^{-3}$ and $\mu = 100$.*

- $B\omega(2\mu+3) = 10^{-3}\cdot 203 = 0.203.$
- $e^{-1} \approx 0.3679$. *Since $B = 0.203 < e^{-1}$, we have $\Lambda^* = -Be\log B/4$. Compute: $\log B = \log 0.203 \approx -1.5946$. Hence $\Lambda^* \approx 0.220$, and $\Lambda^*/2 \approx 0.110$.*
- *$\Lambda_{G>0}$ attains its minimum at $\sigma_* = -1/\log(4\omega) \approx 0.1811$. At $\sigma = \sigma_*$: $\Lambda_{G>0}(\sigma_*) \approx 0.0300$, $\Lambda_{\mathrm{gap}}^{\mathrm{self}}(\sigma_*) \approx 0.374$, $\Lambda_{\mathrm{gap}}(\sigma_*) = \min(0.374, 0.030) = 0.030$.*

Hence $\Lambda^*/2 \approx 0.110 > 0.030 = \Lambda_{\mathrm{gap}}(\sigma_*)$, and the chain fails.

Why this does not affect any result.

- *Λ^* appears only in the definition above; it is not used in the main results.*
- *Λ_{gap} appears only in the hypothesis of Proposition 3.5.*
- *The sharp hypotheses in Sections 4.3–4.4 require $|\lambda| \le \Lambda_{\mathrm{gap}}/2$.*
- *No single result in the paper requires $|\lambda| \le \Lambda^*/2$ together with $|\lambda| \le \Lambda_{\mathrm{gap}}/2$.*

Important clarification. *The statement "no single result requires both conditions together" does not mean that both are unnecessary. A reader wishing to apply all results simultaneously needs both, and the unified domain is $|\lambda| \le \min(\Lambda^*/2, \Lambda_{\mathrm{gap}}/2)$.*

Remark 2.12 (Unified recommended domain for λ)
For a reader wishing to apply all results simultaneously:

$$\boxed{|\lambda| \le \min\left(\frac{\Lambda^*}{2}, \frac{\Lambda_{\mathrm{gap}}}{2}\right).} \tag{45}$$

The detailed domains are given in Table 2.

Table[2]: Required domain on $|\lambda|$ for each result.

Result	Required domain on $\lvert\lambda\rvert$
Proposition 3.5 (self-adjointness)	Λ_{gap}
Theorem 3.1 (relative boundedness of $\mathcal{O}_0^{\sigma}$)	Λ_{gap}
Sections 4.3–4.4 (fixed point)	$\Lambda_{\mathrm{gap}}/2$

Interpretation. *If $\mu \le \mu_c$: $\Lambda_{\mathrm{gap}} = \Lambda_{\mathrm{gap}}^{\mathrm{self}}$, so the minimum is $\Lambda^*/2$. If $\mu > \mu_c$: it may happen that $\Lambda^*/2 > \Lambda_{\mathrm{gap}}/2$, so the minimum is $\Lambda_{\mathrm{gap}}/2$. The domain $\min(\Lambda^*/2, \Lambda_{\mathrm{gap}}/2)$ satisfies all conditions simultaneously.*

2.1.5. Dependence of the Constants on ω

Convention. In the table below, dependence on ω is assessed with $|\gamma|$ and λ fixed. The column "Explicit value" gives the formula when $|\gamma| = \omega/175$ (sharp value).

Key remark. Constants derived from the fixed-point structure ($\|B_n\|$, $\mathrm{Lip}(\Phi_n)$) are independent of ω thanks to the calibration $\gamma_0^{\mathrm{sharp}} = \omega/175$, while constants derived from the gap ($\|A_n\|$, C_{dec}, L_u, $\|\Phi_n\|_\infty$) scale inversely with ω.

Derivation of Eq. (36). Define $f(\sigma)B^{1-\sigma}/(4\sigma)$. Then

$$\log f(\sigma) = (1-\sigma)\log B - \log(4\sigma), \tag{46}$$

whose derivative is

$$\frac{d}{d\sigma}\log f(\sigma) = -\log B - \frac{1}{\sigma}. \tag{47}$$

The unique critical point is $\sigma_c = -1/\log B$. It lies in $(0,1)$ if and only if $\log B < -1$, i.e. $B < e^{-1}$. At this point,

$$\Lambda^* = f(\sigma_c) = \frac{B^{1-\sigma_c}}{4\sigma_c} = \frac{B\cdot B^{-\sigma_c}}{4\sigma_c}. \tag{48}$$

Since $B^{-\sigma_c} = e^{-\sigma_c \log B} = e^1 = e$ (because $\sigma_c \log B = -1$), we obtain

$$\Lambda^* = \frac{Be}{4\sigma_c} = \frac{Be\cdot(-\log B)}{4} = -\frac{Be\log B}{4}. \tag{49}$$

For $B \ge e^{-1}$, f is decreasing on $(0,1)$, and its minimum is attained as $\sigma \to 1^-$: $f(1^-) = B^0/4 = 1/4$.

2.2. Spectrum and Eigenfunctions of the Base Operator

2.2.1. Spectral Analysis

Theorem 2.13 (Spectral analysis of $\mathcal{O}_0$) *The operator $\mathcal{O}_0$ is self-adjoint on $\mathcal{D}(\mathcal{O}_0)$, has purely discrete spectrum, and its orthonormal eigenfunctions and eigenval-*

Table[3]: Dependence of the constants on ω (with $|\gamma| = \omega/175$). Constants are grouped by category: basic constants, gap thresholds, logarithmic thresholds, Schur constants, convolution constants, and fixed-point constants.

Constant	**Scaling in ω**	**Explicit value**
Basic constants (notational shortcuts)		
c_ω	Linear	2ω
R	Linear	$\omega/2$
Gap thresholds		
$\Lambda_{\text{gap}}^{\text{self}}$	$\omega^{1-\sigma}$	$[\omega(2\mu+3)]^{1-\sigma}/(4\sigma)$
$\Lambda_{G>0}$	$\omega^{\sigma-1}$	$1/[2\sigma(4\omega)^{\sigma-1}]$
Λ^*	Function of $B=\omega(2\mu+3)$	$1/4$ if $B \geq e^{-1}$; $-Be\log B/4$ if $0 < B < e^{-1}$
Logarithmic thresholds		
γ_0^{FP}	Linear	$2\omega/175$
γ_0^{sharp}	Linear	$\omega/175$
γ_0^{gap}	Linear	$\omega/(2C_V\log 3) \approx 0.5288\,\omega$
Schur constants		
C_V (digamma difference)	Independent	0.8607
C_1 (Schur row)	Independent	$5/2 = 2.5$
C_2 (Schur column)	Independent	$6697/1800 \approx 3.7206$
$M_0 = M_2 = \|K\|$ (Schur norm)	Independent	$\leq \sqrt{33485}/60 \approx 3.0498$
Convolution constants		
C_2^{tail} (tail constant)	Independent	2
C_2^B (extended convolution)	Independent	≤ 22.4 (rigorous sharp ≈ 22.2)
$\tilde{S}_2$ (logarithmic sum)	Independent	< 1.80
Fixed-point constants ($n \geq 1$)		
$\|A_n(F)\|_{Z_2'}$ (source)	$\propto |\gamma|/\omega$	$\leq 0.415|\gamma|/(2\omega)$
$\|B_n(F)\|$ (contraction)	Independent	≤ 0.032
C_{dec} (decay)	$\propto |\gamma|/\omega$	$\leq 0.429|\gamma|/(2\omega)$
L_u (Lipschitz)	$\propto 1/\omega$	$5.74 \times 10^{-4}/(2\omega)$
$\|\Phi_n\|_\infty$ (consistency)	$\propto |\gamma|^2/\omega$	$\leq 0.772|\gamma|^2/(2\omega)$
$\text{Lip}(\Phi_n)$	Independent	$\leq 2.95 \times 10^{-6}$
$|F_n|$ (correction size)	$\propto \omega$	$\leq 1.26 \times 10^{-5}\omega$
Fixed-point constants ($n = 0$)		
$\|A_0(F)\|_{Z_2'}$ (source)	$\propto |\gamma|/\omega$	$\leq 0.251|\gamma|/(2\omega)$
$\|B_0(F)\|$ (contraction)	Independent	≤ 0.032
$L_u^{(0)}$ (Lipschitz)	$\propto 1/\omega$	$6.7 \times 10^{-4}/(2\omega)$
$\text{Lip}(\Phi_0)$	Independent	$\leq 3.45 \times 10^{-6}$
$|F_0|$ (correction size)	$\propto \omega$	$\leq 1.26 \times 10^{-5}\omega$
Higher-order corrections		
$\gamma^2 E_n^{(2)}$ (2nd order)	$\propto \gamma^2/\omega$	$\leq 0.172\gamma^2/\omega$
$\gamma^3 E_n^{(3)}$ (3rd order)	$\propto |\gamma|^3/\omega^2$	$\leq 0.53|\gamma|^3/\omega^2$ (not recomputed)

ues are

$$\boxed{\begin{aligned}\psi_n(x) =& \mathcal{N}_n x^{\alpha+1/2} e^{-\omega x^2/2} L_n^{(\alpha)}(\omega x^2),\\ & A_n = 4\omega n + \omega(2\mu+3), \quad n \geq 0,\end{aligned}} \tag{50}$$

where $\mathcal{N}_n \sqrt{2\omega^{\alpha+1} n!/\Gamma(n+\alpha+1)}$.

We proceed in five steps.

(1) Self-adjointness. We record the self-adjointness of the maximal operator $\mathcal{O}_0^{\max}$ associated with the Sturm–Liouville expression Eq. (19). This is the standard conclusion of Weyl–Titchmarsh theory for a Sturm–Liouville operator on a half-line with two limit-point endpoints (see, e.g., [21], [23]): at $x = 0^+$, the indicial equation Eq. (22) has roots $s_1 = \mu + 1$ and $s_2 = -\mu$, and since $\mu > 1/2$ the singular solution $x^{-\mu}$ is not in $L^2(0, \epsilon)$; at $x = \infty$, the potential $\omega^2 x^2 \to \infty$ forces a limit-point endpoint. A self-contained verification of both limit-point endpoints is given in Lemma 2.25 below. By the Weyl–Titchmarsh theorem, the maximal operator $\mathcal{O}_0^{\max}$ is self-adjoint, and it is the unique self-adjoint extension of $A_0 = \mathcal{O}_0|_{C_c^\infty(0,\infty)}$. A proof that the domain $\mathcal{D}(\mathcal{O}_0)$ of Definition 2.2 coincides with $\mathcal{D}(\mathcal{O}_0^{\max})$ is given in Theorem 2.30 below; the combination of the two results then shows that $\mathcal{O}_0$ on $\mathcal{D}(\mathcal{O}_0)$ is self-adjoint. To avoid any appearance of circularity, we emphasize that Steps 2–5 of the present theorem are purely constructive (Laguerre reduction, normalization, completeness) and do not rely on self-adjointness.

(2) Reduction to Laguerre's equation. We present the full derivation in five steps. Set $y\omega x^2$ and write

$$\psi(x) = x^{\alpha+1/2}E(x)v(y), \qquad E(x)e^{-\omega x^2/2}. \tag{51}$$

Note that $dy/dx = 2\omega x$, hence $d/dx = 2\omega x \cdot d/dy$.

Step 1 (first derivative). By the product and chain rules,

$$\psi'(x) = \frac{d}{dx}[x^{\alpha+1/2}]Ev + x^{\alpha+1/2}\frac{dE}{dx}v + x^{\alpha+1/2}E\frac{dv}{dx}. \tag{52}$$

We compute each term separately:

$$\frac{d}{dx}[x^{\alpha+1/2}] = (\alpha+\tfrac{1}{2})x^{\alpha-1/2};$$

$$\frac{dE}{dx} = \frac{d}{dx}[e^{-\omega x^2/2}] = -\omega x e^{-\omega x^2/2} = -\omega x E;$$

$$\frac{dv}{dx} = \frac{dv}{dy}\cdot\frac{dy}{dx} = 2\omega x v'(y).$$

Substituting into Eq. (52):

$$\begin{aligned}\psi'(x) =&(\alpha+\tfrac{1}{2})x^{\alpha-1/2}Ev - \omega x\cdot x^{\alpha+1/2}Ev\\ &+ x^{\alpha+1/2}E\cdot 2\omega x v'.\end{aligned} \tag{53}$$

Since $y = \omega x^2$:

$$\boxed{\psi'(x) = x^{\alpha-1/2}E\left[(\alpha+\tfrac{1}{2})v - yv + 2yv'\right].} \tag{54}$$

Step 2 (second derivative). Write $\psi'(x) = g(x)h(x)$, where

$$\begin{aligned}g(x)x^{\alpha-1/2}, \quad h(x)E(x)G(y),\\ G(y)(\alpha+\tfrac{1}{2})v - yv + 2yv'.\end{aligned} \tag{55}$$

Then $\psi''(x) = g'(x)h(x) + g(x)h'(x)$. We compute:

$$g'(x) = (\alpha-\tfrac{1}{2})x^{\alpha-3/2}; \tag{56}$$

$$\begin{aligned}h'(x) =&\frac{dE}{dx}G + E\frac{dG}{dx} = -\omega x EG + 2\omega x EG_y,\\ &\text{where } G_y dG/dy.\end{aligned} \tag{57}$$

Step 3 (explicit expansion of G_y and ψ''). We compute G_y by differentiating each term of G:

$$\begin{aligned}G_y &= \frac{d}{dy}\left[(\alpha+\tfrac{1}{2})v - yv + 2yv'\right]\\ &= (\alpha+\tfrac{1}{2})v' - (v + yv') + (2v' + 2yv'')\\ &= (\alpha+\tfrac{1}{2})v' - v - yv' + 2v' + 2yv''\\ &= (\alpha+\tfrac{1}{2}+2)v' - yv' - v + 2yv''.\end{aligned} \tag{58}$$

Since $\alpha + \frac{1}{2} + 2 = \alpha + \frac{5}{2}$:

$$\boxed{G_y = (\alpha+\tfrac{5}{2}-y)v' - v + 2yv''.} \tag{59}$$

Now we compute $\psi''(x) = g'h + gh'$:

$$g'h = (\alpha-\tfrac{1}{2})x^{\alpha-3/2}EG; \tag{60}$$

$$\begin{aligned}gh' =&x^{\alpha-1/2}E[-\omega x G + 2\omega x G_y] =\\ &\omega x^{\alpha+1/2}E[-G+2G_y].\end{aligned} \tag{61}$$

Factoring out $x^{\alpha-3/2}E$:

$$\psi''(x) = x^{\alpha-3/2}E\left[(\alpha-\tfrac{1}{2})G + \omega x^2(-G+2G_y)\right]. \tag{62}$$

Since $\omega x^2 = y$, we have $\omega x^2(-G+2G_y) = -yG + 2yG_y$. Hence

$$\psi''(x) = x^{\alpha-3/2}E\left[(\alpha-\tfrac{1}{2}-y)G + 2yG_y\right]. \tag{63}$$

We now substitute G and G_y and collect by powers of v, v', v''.

(3a) Contribution of $(\alpha-\frac{1}{2}-y)G$. Recall $G = (\alpha+\frac{1}{2})v - yv + 2yv'$. Then

$$(\alpha-\tfrac{1}{2}-y)G = (\alpha-\tfrac{1}{2}-y)[(\alpha+\tfrac{1}{2})v - yv + 2yv']. \tag{64}$$

We expand term by term:

For v from $(\alpha+\frac{1}{2})v$:

$$(\alpha-\tfrac{1}{2})(\alpha+\tfrac{1}{2})v - y(\alpha+\tfrac{1}{2})v = (\alpha^2-\tfrac{1}{4})v - y(\alpha+\tfrac{1}{2})v.$$

For v from $-yv$:

$$-y(\alpha-\tfrac{1}{2})v + y^2 v.$$

For v' from $2yv'$:

$$2y(\alpha-\tfrac{1}{2})v' - 2y^2v'.$$

Summing all v-terms:

$$\begin{aligned}\mathrm{coeff}_v &= (\alpha^2-\tfrac{1}{4}) - y(\alpha+\tfrac{1}{2}) - y(\alpha-\tfrac{1}{2}) + y^2\\ &= (\alpha^2-\tfrac{1}{4}) - y[(\alpha+\tfrac{1}{2}) + (\alpha-\tfrac{1}{2})] + y^2\\ &= (\alpha^2-\tfrac{1}{4}) - 2\alpha y + y^2.\end{aligned} \tag{65}$$

Coefficient of v': $2y(\alpha-\frac{1}{2}) - 2y^2$.

(3b) Contribution of $2yG_y$. Recall $G_y = (\alpha+\frac{5}{2}-y)v' - v + 2yv''$. Then:

$$2y\cdot(\alpha+\tfrac{5}{2}-y)v' = 2y(\alpha+\tfrac{5}{2})v' - 2y^2v';$$

$$2y\cdot(-v) = -2yv;$$

$$2y\cdot 2yv'' = 4y^2v''.$$

(3c) Total coefficient. Combining (3a) and (3b):

$$\begin{aligned}\mathrm{coeff}_{v''} &= 4y^2,\\ \mathrm{coeff}_{v'} &= 2y(\alpha-\tfrac{1}{2}) - 2y^2 + 2y(\alpha+\tfrac{5}{2}) - 2y^2\\ &= 2y[(\alpha-\tfrac{1}{2}) + (\alpha+\tfrac{5}{2})] - 4y^2\\ &= 2y(2\alpha+2) - 4y^2\\ &= 4y(\alpha+1) - 4y^2 = 4y(\alpha+1-y),\\ \mathrm{coeff}_v &= (\alpha^2-\tfrac{1}{4}) - 2\alpha y + y^2 - 2y\\ &= (\alpha^2-\tfrac{1}{4}) - 2(\alpha+1)y + y^2.\end{aligned} \tag{66}$$

Hence

$$\boxed{\begin{aligned}\psi''(x) = x^{\alpha-3/2}E\Big[&4y^2v'' + 4y(\alpha+1-y)v' +\\ &\Big((\alpha^2-\tfrac{1}{4}) - 2(\alpha+1)y + y^2\Big)v\Big].\end{aligned}} \tag{67}$$

Step 4 (substitution in the eigenvalue equation). The eigenvalue equation is

$$-\psi'' + \left[\frac{\alpha^2 - 1/4}{x^2} + \omega^2 x^2\right]\psi = A\psi, \tag{68}$$

where we used $\mu(\mu+1) = \alpha^2 - 1/4$. First, we compute the potential term:

$$\begin{aligned}\left[\frac{\alpha^2 - 1/4}{x^2} + \omega^2 x^2\right]\psi = x^{\alpha-3/2}E[(\alpha^2 - 1/4)v + \\ \omega^2 x^4 v] = x^{\alpha-3/2}E[(\alpha^2 - 1/4)v + y^2 v],\end{aligned} \tag{69}$$

since $\omega^2 x^4 = (\omega x^2)^2 = y^2$. Also

$$A\psi = x^{\alpha-3/2}E \cdot (A/\omega)yv, \tag{70}$$

since $A\psi = A \cdot x^{\alpha+1/2}Ev = x^{\alpha-3/2}E \cdot A(x^2/\omega)v = x^{\alpha-3/2}E \cdot (A/\omega)yv$.

Substituting Eq. (67), Eq. (69), and Eq. (70) into Eq. (68):

$$\begin{aligned}-\Big[4y^2 v'' + 4y(\alpha+1-y)v' + ((\alpha^2 - \tfrac{1}{4}) - \\ 2(\alpha+1)y + y^2)v\Big] + (\alpha^2 - \tfrac{1}{4})v + y^2 v = \frac{A}{\omega}yv.\end{aligned} \tag{71}$$

We simplify by combining the v-terms:

$$\begin{aligned}\mathrm{LHS}_v = -[(\alpha^2 - \tfrac{1}{4}) - 2(\alpha+1)y + y^2] + (\alpha^2 - \tfrac{1}{4}) \\ + y^2 = -(\alpha^2 - \tfrac{1}{4}) + 2(\alpha+1)y - y^2 \\ + (\alpha^2 - \tfrac{1}{4}) + y^2 = 2(\alpha+1)y.\end{aligned} \tag{72}$$

Hence the full equation becomes

$$-4y^2 v'' - 4y(\alpha+1-y)v' + 2(\alpha+1)yv = \frac{A}{\omega}yv. \tag{73}$$

We divide by $-4y$ (for $y > 0$):

$$yv'' + (\alpha+1-y)v' - \frac{(\alpha+1)}{2}v = -\frac{A}{4\omega}v. \tag{74}$$

Equivalently, moving the v-term to the right:

$$\boxed{\begin{aligned}yv''(y) + (\alpha+1-y)v'(y) + \\ \left[\frac{A}{4\omega} - \frac{\alpha+1}{2}\right]v(y) = 0.\end{aligned}} \tag{75}$$

Step 5 (identification with Laguerre's equation). Laguerre's equation is

$$yL_n^{(\alpha)\prime\prime}(y) + (\alpha+1-y)L_n^{(\alpha)\prime}(y) + nL_n^{(\alpha)}(y) = 0, \tag{76}$$

with polynomial solutions $L_n^{(\alpha)}$ of degree n for $n \in \mathbb{Z}_{\geq 0}$ and $\alpha > -1$. Comparing Eq. (75) with Eq. (76) term by term:

$$n = \frac{A}{4\omega} - \frac{\alpha+1}{2} = \frac{A}{4\omega} - \frac{2\mu+3}{4}. \tag{77}$$

From $n = A/(4\omega) - (2\mu+3)/4$, we solve for A:

$$A = 4\omega n + \omega(2\mu+3). \tag{78}$$

Hence $A_n = 4\omega n + \omega(2\mu+3)$.

Existence of solutions. Since $\alpha = \mu + 1/2 > 1 > -1$ (because $\mu > 1/2$), the conditions for existence of $L_n^{(\alpha)}$ are satisfied.

(3) Normalization. We use the orthogonality relation for generalized Laguerre polynomials:

$$\int_0^\infty y^\alpha e^{-y} L_m^{(\alpha)}(y)L_n^{(\alpha)}(y)\,dy = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}. \tag{79}$$

Verification of the normalization constant. We compute $\|\psi_n\|^2 = \int_0^\infty |\psi_n(x)|^2\,dx$. Under $y = \omega x^2$:

$$|\psi_n(x)|^2 = \mathcal{N}_n^2 (y/\omega)^{\alpha+1/2} e^{-y}(L_n^{(\alpha)}(y))^2. \tag{80}$$

Since $dx = \frac{1}{2}\omega^{-1/2}y^{-1/2}dy$, we have

$$\begin{aligned}&\|\psi_n\|^2 = \\ &\mathcal{N}_n^2 \int_0^\infty (y/\omega)^{\alpha+1/2} e^{-y}(L_n^{(\alpha)}(y))^2 \cdot \frac{1}{2}\omega^{-1/2}y^{-1/2}\,dy \\ &= \frac{\mathcal{N}_n^2}{2}\omega^{-\alpha-1}\int_0^\infty y^\alpha e^{-y}(L_n^{(\alpha)}(y))^2\,dy \\ &= \frac{\mathcal{N}_n^2}{2}\omega^{-\alpha-1} \cdot \frac{\Gamma(n+\alpha+1)}{n!}.\end{aligned} \tag{81}$$

Setting $\|\psi_n\|^2 = 1$:

$$\mathcal{N}_n^2 = \frac{2\omega^{\alpha+1}n!}{\Gamma(n+\alpha+1)}, \tag{82}$$

and hence

$$\boxed{\mathcal{N}_n = \sqrt{\frac{2\omega^{\alpha+1}n!}{\Gamma(n+\alpha+1)}}.} \tag{83}$$

(4) Completeness. We prove it in four steps.

(4a) Compactness of the resolvent. The potential $V(x) \to +\infty$ at both endpoints. This is the Rellich property for operators with confining potentials: the embedding $\mathcal{D}(q_0) \hookrightarrow L^2$ is compact (see Remark 2.18 and the quadratic form defined in Eq. (91) for the precise meaning of $\mathcal{D}(q_0)$).

(4b) Discrete spectrum. From (4a), $(\mathcal{O}_0 + i)^{-1}$ is compact. By the spectral theorem, the spectrum consists of discrete eigenvalues A_n with eigenfunctions ψ_n.

(4c) Orthogonality. The family $\{\psi_n\}_{n\geq 0}$ is orthonormal by Eq. (79).

(4d) Completeness. Assume $f \in L^2$ satisfies $\langle f, \psi_n\rangle = 0$ for every $n \geq 0$. Under the substitution $y = \omega x^2$, the pairing $\langle f, \psi_n\rangle$ becomes (up to a positive constant)

$$\int_0^\infty \tilde{f}(y)L_n^{(\alpha)}(y)y^\alpha e^{-y}\,dy, \tag{84}$$

where

$$\tilde{f}(y)f((y/\omega)^{1/2})y^{-\alpha/2-1/4}e^{y/2}. \tag{85}$$

We verify that $\tilde{f} \in L^2((0,\infty), y^\alpha e^{-y}dy)$ by direct computation:

$$\begin{aligned}&\int_0^\infty |\tilde{f}(y)|^2 y^\alpha e^{-y}\,dy \\ &= \int_0^\infty |f((y/\omega)^{1/2})|^2 y^{-\alpha-1/2}e^y \cdot y^\alpha e^{-y}\,dy \\ &= \int_0^\infty |f((y/\omega)^{1/2})|^2 y^{-1/2}\,dy.\end{aligned} \tag{86}$$

Substituting $u = (y/\omega)^{1/2}$, so $y = \omega u^2$ and $dy =$

$2\omega u\,du$:

$$\text{Eq. (86)} = \int_0^\infty |f(u)|^2(\omega u^2)^{-1/2}\cdot 2\omega u\,du$$
$$= 2\omega^{1/2}\int_0^\infty |f(u)|^2\,du = 2\omega^{1/2}\|f\|^2 < \infty. \quad (87)$$

By assumption, $\tilde{f}$ is orthogonal to every $L_n^{(\alpha)}$. By completeness of generalized Laguerre polynomials in $L^2((0,\infty), y^\alpha e^{-y}dy)$ for $\alpha > -1$, we have $\tilde{f} = 0$ a.e. Hence $f = 0$ a.e., and $\{\psi_n\}_{n\geq 0}$ is complete.

(5) No other eigenvalues. Since $\{\psi_n\}_{n\geq 0}$ is a complete orthonormal basis and $\mathcal{O}_0$ is self-adjoint, the spectral theorem gives the representation

$$\mathcal{O}_0 = \sum_{n\geq 0} A_n\langle\cdot,\psi_n\rangle\psi_n, \quad (88)$$

so $\sigma(\mathcal{O}_0) = \overline{\{A_n : n\geq 0\}} = \{A_n : n \geq 0\}$. In particular, there are no other eigenvalues.

Lemma 2.14 (Compactness of the resolvent) *The resolvent $(\mathcal{O}_0 + i)^{-1}$ is compact on $L^2(0,\infty)$.*

In the orthonormal basis $\{\psi_n\}_{n\geq 0}$ obtained in Theorem 2.13, the resolvent acts diagonally:

$$(\mathcal{O}_0 + i)^{-1}\psi_n = (A_n + i)^{-1}\psi_n. \quad (89)$$

The singular values are

$$\sigma_n|A_n + i|^{-1} = \frac{1}{\sqrt{A_n^2+1}}. \quad (90)$$

Since $A_n \to +\infty$ as $n\to\infty$, we have $\sigma_n \to 0$. By the characterization of singular values of compact operators, $(\mathcal{O}_0 + i)^{-1}$ is compact.

Corollary 2.15 (Absence of essential spectrum) $\sigma_{\text{ess}}(\mathcal{O}_0) = \emptyset$.

By Lemma 2.14, the resolvent is compact. For a self-adjoint operator, compactness of the resolvent at a point $z \notin \sigma(\mathcal{O}_0)$ is equivalent to $\sigma_{\text{ess}}(\mathcal{O}_0) = \emptyset$.

Remark 2.16 *The lowest eigenvalue is $A_0 = \omega(2\mu+3) > 0$.*

2.2.2. Quadratic Form and Interpolation Inequality

The quadratic form. Define

$$q_0(f)\int_0^\infty \left[|f'|^2 + \frac{\mu(\mu+1)}{x^2}|f|^2 + \omega^2x^2|f|^2\right]dx, \quad (91)$$

with domain

$$\mathcal{D}(q_0)\left\{f\in L^2(0,\infty) : f'\in L^2,\right.$$
$$\left. f/x \in L^2,\ xf\in L^2\right\}. \quad (92)$$

Lemma 2.17 (Closedness of q_0) *The quadratic form q_0 is non-negative and closed on $\mathcal{D}(q_0)$, and densely defined: $C_c^\infty(0,\infty)$ is dense in $\mathcal{D}(q_0)$ with respect to the form norm.*

Non-negativity. $q_0(f) = \int_0^\infty(|f'|^2 + V|f|^2)dx \geq 0$.
Closedness. Equip $\mathcal{D}(q_0)$ with the form norm

$$\|f\|_{q_0}^2 q_0(f) + \|f\|_{L^2}^2. \quad (93)$$

We write

$$\mathcal{D}(q_0) = H_1\cap H_2\cap H_3, \quad (94)$$

where

$$H_1 H^1(0,\infty),\quad H_2\{f\in L^2 : f/x\in L^2\},$$
$$H_3\{f\in L^2 : xf\in L^2\}. \quad (95)$$

Each is a Hilbert space with the natural graph norm:

$$\|f\|_{H_1}^2\|f'\|^2 + \|f\|^2,\quad \|f\|_{H_2}^2\|f/x\|^2 + \|f\|^2,$$
$$\|f\|_{H_3}^2\|xf\|^2 + \|f\|^2. \quad (96)$$

The maps $f\mapsto(f, f/x)$ and $f\mapsto(f,xf)$ are linear isometries from H_2, H_3 onto closed subspaces of $L^2\times L^2$. Indeed, $\|\Phi_2(f)\|_{L^2\times L^2}^2 = \|f\|^2 + \|f/x\|^2 = \|f\|_{H_2}^2$, and closedness of $\Phi_2(H_2)$ follows from: a convergent sequence $f_n \to f$ in L^2 with $f_n/x \to g$ in L^2 forces $g = f/x$ distributionally, hence $f\in H_2$. The same argument applies to Φ_3.
Since $\mu(\mu+1) > 0$ and $\omega^2 > 0$, the form norm and the intersection norm

$$\|f\|_{\text{int}}^2\|f\|^2 + \|f'\|^2 + \|f/x\|^2 + \|xf\|^2 \quad (97)$$

are equivalent:

$$\boxed{c_-\|f\|_{\text{int}}^2 \leq \|f\|_{q_0}^2 \leq c_+\|f\|_{\text{int}}^2,} \quad (98)$$

with $c_-\min(1,\mu(\mu+1),\omega^2) > 0$ and $c_+\max(1,\mu(\mu+1),\omega^2) < \infty$. This proves that $\mathcal{D}(q_0)$ is complete with respect to $\|\cdot\|_{q_0}$, so q_0 is closed.
Density. We show that $C_c^\infty(0,\infty)$ is dense in $\mathcal{D}(q_0)$.
(i) *Truncation.* For every small $\epsilon > 0$ and large $N > 1$, choose $\chi_{\epsilon,N}\in C_c^\infty(0,\infty)$ with $0\leq\chi_{\epsilon,N}\leq 1$, $\chi_{\epsilon,N} = 1$ on $[\epsilon, N]$, $|\chi'_{\epsilon,N}|\leq K/\epsilon$ on $(0,\epsilon)$, and $|\chi'_{\epsilon,N}|\leq K/N$ on (N,∞), where $K > 0$ is a universal constant (independent of ϵ and N). Set $f_{\epsilon,N}\chi_{\epsilon,N}f$. Then $f_{\epsilon,N}\to f$ in $\mathcal{D}(q_0)$ as $\epsilon\to 0^+$, $N\to\infty$, by dominated convergence applied to $|f|^2, |f'|^2, |f/x|^2, x^2|f|^2$. Indeed, for the small-$x$ contribution we have: on $(0,\epsilon)$, $x\leq\epsilon$ implies $x^2\leq\epsilon^2$, hence $1/x^2 \geq 1/\epsilon^2$, and therefore

$$\int_0^\epsilon |\chi'_{\epsilon,N}|^2|f|^2\,dx \leq \frac{K^2}{\epsilon^2}\int_0^\epsilon |f|^2\,dx \leq K^2$$
$$\int_0^\epsilon \frac{|f|^2}{x^2}\,dx \xrightarrow{\epsilon\to 0^+} 0, \quad (99)$$

since $f/x\in L^2(0,\infty)$. The large-x contribution vanishes by a similar argument using $xf\in L^2(N,\infty)$.
(ii) *Mollification.* Let ρ_δ be a standard mollifier. Since $f_{\epsilon,N}$ has compact support in $[\epsilon/2, 2N]$, the convolution $f_{\epsilon,N,\delta}\rho_\delta * f_{\epsilon,N}$ lies in $C_c^\infty(0,\infty)$ for $\delta < \epsilon/4$ (its support is contained in $[\epsilon/4, 2N+\delta]$, compact in $(0,\infty)$). By standard mollification properties, $f_{\epsilon,N,\delta}\to f_{\epsilon,N}$ in $\mathcal{D}(q_0)$ as $\delta\to 0$.

Remark 2.18 (Friedrichs extension) *By the Friedrichs extension theorem, since q_0 is a densely defined, closed, lower-semibounded quadratic form on $L^2(0,\infty)$, it admits a unique self-adjoint operator $\mathcal{O}_0^F$ with $\mathcal{D}(\mathcal{O}_0^F) \subseteq \mathcal{D}(q_0)$ and $q_0(f,g) = \langle \mathcal{O}_0^F f, g\rangle$ for every $f \in \mathcal{D}(\mathcal{O}_0^F)$, $g \in \mathcal{D}(q_0)$.*

Extension check. *Integration by parts on $C_c^\infty(0,\infty)$ shows $q_0(\phi,g) = \langle \mathcal{O}_0\phi, g\rangle$ for every $\phi \in C_c^\infty(0,\infty)$, $g \in \mathcal{D}(q_0)$. By the Friedrichs construction, $\phi \in \mathcal{D}(\mathcal{O}_0^F)$ and $\mathcal{O}_0^F\phi = \mathcal{O}_0\phi = A_0\phi$. Hence $\mathcal{O}_0^F \supseteq A_0\mathcal{O}_0|_{C_c^\infty(0,\infty)}$.*
The exact identification of $\mathcal{O}_0^F$ with $\mathcal{O}_0$ (that is, $\mathcal{O}_0^F = \mathcal{O}_0$ including domains) is given in Theorem 2.30 below, where Corollary 2.26(ii) is used.

Lemma 2.19 (Minimum of the potential) *The potential $V(x)\mu(\mu+1)/x^2 + \omega^2x^2$ attains its minimum at $x_*[\mu(\mu+1)/\omega^2]^{1/4}$, with*

$$\boxed{c_0 \min_{x>0} V(x) = 2\omega\sqrt{\mu(\mu+1)}.} \tag{100}$$

Consequently, $q_0(f) \geq c_0\|f\|^2$.

We compute $V'(x) = -2\mu(\mu+1)/x^3 + 2\omega^2 x$. Setting $V'(x) = 0$:

$$2\omega^2 x = \frac{2\mu(\mu+1)}{x^3}, \quad \text{so} \quad x^4 = \frac{\mu(\mu+1)}{\omega^2}, \quad x_* = \left[\frac{\mu(\mu+1)}{\omega^2}\right]^{1/4}. \tag{101}$$

The second derivative is $V''(x) = 6\mu(\mu+1)/x^4 + 2\omega^2 > 0$, so x_* is a strict global minimum.
We compute $V(x_*)$. First, $x_*^2 = [\mu(\mu+1)/\omega^2]^{1/2} = \sqrt{\mu(\mu+1)}/\omega$. Hence:

$$\frac{\mu(\mu+1)}{x_*^2} = \mu(\mu+1)\cdot\omega/\sqrt{\mu(\mu+1)} = \omega\sqrt{\mu(\mu+1)};$$

$$\omega^2x_*^2 = \omega^2\cdot\sqrt{\mu(\mu+1)}/\omega = \omega\sqrt{\mu(\mu+1)}.$$

Sum: $2\omega\sqrt{\mu(\mu+1)}$. Since $V(x) \to +\infty$ at both endpoints, x_* is the global minimum.

Lemma 2.20 (Interpolation inequality) *For every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$q_0(f) \leq \epsilon\|\mathcal{O}_0 f\|^2 + C_\epsilon\|f\|^2, \qquad \forall f \in \mathcal{D}(\mathcal{O}_0). \tag{102}$$

Moreover, with $C = 1$:

$$q_0(f) \leq \|\mathcal{O}_0 f\|\|f\|. \tag{103}$$

In the eigenbasis $f = \sum_n c_n\psi_n$ (which is available by Theorem 2.13):

$$q_0(f) = \sum_n A_n|c_n|^2, \quad \|\mathcal{O}_0 f\|^2 = \sum_n A_n^2|c_n|^2, \quad \|f\|^2 = \sum_n |c_n|^2. \tag{104}$$

For each $n \geq 0$, the inequality $A_n \leq \epsilon A_n^2 + C_\epsilon$ holds with $C_\epsilon 1/(4\epsilon)$, because the discriminant of $\epsilon t^2 - t + C_\epsilon$ vanishes: $\Delta = 1 - 4\epsilon\cdot 1/(4\epsilon) = 0$. Hence $\epsilon t^2 - t + C_\epsilon \geq 0$ for all t. Summing:

$$q_0(f) = \sum_n A_n|c_n|^2 \leq \sum_n(\epsilon A_n^2 + C_\epsilon)|c_n|^2 = \epsilon\|\mathcal{O}_0 f\|^2 + C_\epsilon\|f\|^2. \tag{105}$$

For the second inequality, use Cauchy–Schwarz:

$$q_0(f) = \langle \mathcal{O}_0 f, f\rangle = \sum_n A_n|c_n|^2 \leq \left(\sum_n A_n^2|c_n|^2\right)^{1/2} \left(\sum_n |c_n|^2\right)^{1/2} = \|\mathcal{O}_0 f\|\|f\|. \tag{106}$$

Remark 2.21 (Hardy's inequality) *The classical Hardy inequality on the half-line states:*

$$\int_0^\infty \frac{|f(x)|^2}{x^2}\,dx \leq 4\int_0^\infty |f'(x)|^2\,dx, \qquad f \in H_0^1(0,\infty), \tag{107}$$

where $H_0^1(0,\infty)\overline{C_c^\infty(0,\infty)}^{\|\cdot\|_{H^1}}$. The essential condition is that f "vanishes" at $x = 0$ in the trace sense ($f(0) = 0$). Without this condition, the inequality fails: for example, $f(x) = e^{-x} \in H^1(0,\infty)$ gives $\int_0^\infty e^{-2x}/x^2\,dx = \infty$, while $\int_0^\infty |f'|^2 < \infty$.
General form (with boundary value subtracted). *For every $f \in H^1(0,\infty)$,*

$$\int_0^\infty \frac{|f(x) - f(0)|^2}{x^2}\,dx \leq 4\int_0^\infty |f'(x)|^2\,dx, \tag{108}$$

where $f(0)$ is understood as the trace of f at $x = 0$ (a continuous functional on $H^1(0,\infty)$).

Relevance to $\mathcal{D}(q_0)$. *In the definition of $\mathcal{D}(q_0)$, the condition $f/x \in L^2$ is imposed explicitly. This condition implies (in the distributional sense) that $f(0) = 0$, so inequality* Eq. (107) *applies. Hence $\mathcal{D}(q_0)$ is a well-defined Hilbert space, and the term f/x is controlled by f' via* Eq. (107).

Reference: [36, Theorem 327].

2.2.3. Minimal and Maximal Operators, and Maximal Domain

Organizational note. The detailed proof of the limit-point property is given below in Lemma 2.25. The reader may consult it before or after reading Theorem 2.13, since both use only the elementary indicial equation $s(s-1) = \mu(\mu+1)$ established in Remark 2.4 and its standard consequences.

Definition 2.22 (Minimal operator) *Define $A_0\mathcal{O}_0|_{C_c^\infty(0,\infty)}$. The minimal operator $\mathcal{O}_0^{\min}$ is the closure of A_0 in $L^2(0,\infty)$ in the graph sense:*

$$\mathcal{O}_0^{\min}\overline{A_0}, \tag{109}$$

where the closure is taken in the graph sense in $L^2(0,\infty) \oplus L^2(0,\infty)$. Its domain is denoted by $\mathcal{D}(\mathcal{O}_0^{\min})$.

Definition 2.23 (Maximal operator) *The maximal operator $\mathcal{O}_0^{\max}$ acts as $\mathcal{O}_0$ on*

$$\mathcal{D}(\mathcal{O}_0^{\max}) \Big\{ f \in L^2(0,\infty) : \exists g \in L^2, \langle f, \mathcal{O}_0\phi \rangle = \langle g, \phi \rangle \ \forall \phi \in C_c^\infty(0,\infty). \tag{110}$$

Then $\mathcal{O}_0^{\max} fg$.

Remark 2.24 (Basic properties of A_0) (a) Symmetry. *A_0 is symmetric on $L^2(0,\infty)$. For $\phi, \psi \in C_c^\infty(0,\infty)$:*

$$\langle A_0\phi, \psi \rangle = \int_0^\infty (-\phi'' + V\phi)\overline{\psi}\, dx. \tag{111}$$

Integration by parts on (ϵ, N) and taking the limits $\epsilon \to 0^+$, $N \to \infty$:

$$\int_0^\infty (-\phi'')\overline{\psi}\, dx = \int_0^\infty \phi'\overline{\psi'}\, dx - [\phi'\overline{\psi}]_0^\infty. \tag{112}$$

The boundary term vanishes because ϕ, ψ have compact support in $(0,\infty)$. Similarly,

$$\int_0^\infty \phi'\overline{\psi'}\, dx = \int_0^\infty (-\phi)\overline{\psi''}\, dx + [\phi\overline{\psi'}]_0^\infty, \tag{113}$$

with vanishing boundary term. Hence

$$\int_0^\infty (-\phi'')\overline{\psi}\, dx = \int_0^\infty \phi\overline{(-\psi'')}\, dx. \tag{114}$$

Since V is real, $\int V\phi\overline{\psi}\, dx = \int \phi\overline{V\psi}\, dx$. Hence

$$\langle A_0\phi, \psi \rangle = \langle \phi, A_0\psi \rangle. \tag{115}$$

Therefore A_0 is closable, and its closure $\mathcal{O}_0^{\min}$ is symmetric.

(b) Containment. *$\mathcal{O}_0^{\min} \subseteq \mathcal{O}_0^{\max}$: by the definition of $\mathcal{O}_0^{\max}$ as the maximal closed operator extending A_0, and since $\mathcal{O}_0^{\min}$ is the graph closure of A_0.*

Lemma 2.25 (Limit-point nature at both endpoints) *For every $\mu > 1/2$ and $\omega > 0$, the Sturm–Liouville problem $\mathcal{O}_0 f = zf$ is of limit-point type at $x = 0^+$ and at $x = \infty$.*

(a) At $x = 0^+$. As established in Remark 2.4, the indicial equation Eq. (22) has roots $s_1 = \mu + 1$, $s_2 = -\mu$ (see Eq. (23)). Since $\mu > 1/2$, we have $-\mu < -1/2$, so $2\mu > 1$. We compute

$$\int_0^\epsilon |x^{-\mu}|^2\, dx = \int_0^\epsilon x^{-2\mu}\, dx = \lim_{a\to 0^+} \frac{\epsilon^{1-2\mu} - a^{1-2\mu}}{1-2\mu} = +\infty, \tag{116}$$

since $1 - 2\mu < 0$ and $a^{1-2\mu} \to +\infty$ as $a \to 0^+$. Meanwhile

$$\int_0^\epsilon x^{2(\mu+1)}\, dx = \frac{\epsilon^{2\mu+3}}{2\mu+3} < \infty. \tag{117}$$

Hence exactly one of the two linearly independent solutions $\{x^{\mu+1}, x^{-\mu}\}$ lies in $L^2(0,\epsilon)$. This is the definition of limit-point at $x = 0$.

(b) At $x = \infty$. Since $V(x) = \mu(\mu+1)/x^2 + \omega^2 x^2 \to +\infty$, the equation $\mathcal{O}_0 f = zf$ has two linearly independent solutions with asymptotic behaviour $e^{\pm\omega x^2/2}(1 + o(1))$, by the WKB (Liouville–Green) approximation for Schrödinger operators with confining potentials (see, e.g., [50] for the standard WKB treatment). Precisely, $\sqrt{V(x)} = \sqrt{\mu(\mu+1)/x^2 + \omega^2 x^2} \sim \omega x$ as $x \to \infty$, so $\int^R \sqrt{V(t)}\, dt = \omega x^2/2 + O(\log x)$, giving the stated asymptotic. Only $e^{-\omega x^2/2}$ lies in $L^2(R,\infty)$ for every $R > 0$:

$$\int_R^\infty |e^{-\omega x^2/2}|^2\, dx = \int_R^\infty e^{-\omega x^2}\, dx \le \int_0^\infty e^{-\omega x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\omega}} < \infty. \tag{118}$$

The second solution gives

$$\int_R^\infty |e^{+\omega x^2/2}|^2\, dx = \int_R^\infty e^{+\omega x^2}\, dx = \infty. \tag{119}$$

Hence exactly one of the two solutions lies in $L^2(R,\infty)$, so the problem is of limit-point type at $x = \infty$.

Corollary 2.26 ($\min = \max$, essential self-adjointness) *For every $\mu > 1/2$ and $\omega > 0$,*

$$\boxed{\mathcal{O}_0^{\min} = \mathcal{O}_0^{\max},} \tag{120}$$

and this common operator is self-adjoint. Moreover:

(i) The symmetric operator $A_0 = \mathcal{O}_0|_{C_c^\infty(0,\infty)}$ is essentially self-adjoint on $L^2(0,\infty)$; its unique self-adjoint extension is $\mathcal{O}_0^{\min}$.
(ii) Every self-adjoint extension B of A_0 satisfies $B = \mathcal{O}_0^{\min} = \mathcal{O}_0^{\max}$ (as operators, including domains).

By Lemma 2.25, the Sturm–Liouville problem is of limit-point type at both endpoints. By the standard theorem for singular Sturm–Liouville operators with limit-point endpoints (see e.g. [23] and [51]), the minimal and maximal operators coincide and this common operator is self-adjoint. This gives Eq. (120) and part (i).

Part (ii). Let B be a self-adjoint extension of A_0: $A_0 \subseteq B$. Taking adjoints: $B^* \subseteq A_0^*$. Since A_0 is essentially self-adjoint and B is self-adjoint, we have $A_0^* = \mathcal{O}_0^{\min}$ and $B^* = B$. Hence $B \subseteq \mathcal{O}_0^{\min}$. Conversely, $\mathcal{O}_0^{\min} = \overline{A_0} \subseteq B$ (the graph closure of A_0 is contained in any closed extension of A_0). Hence $B = \mathcal{O}_0^{\min} = \mathcal{O}_0^{\max}$.

Theorem 2.27 (Identification of the Friedrichs extension) *For every $\mu > 1/2$ and $\omega > 0$, the Friedrichs extension $\mathcal{O}_0^F$ coincides with the minimal and maximal operators:*

$$\boxed{\mathcal{O}_0^F = \mathcal{O}_0^{\min} = \mathcal{O}_0^{\max}.} \tag{121}$$

By Remark 2.18, $\mathcal{O}_0^F$ is a self-adjoint extension of A_0. By Corollary 2.26(ii), every self-adjoint extension of A_0 equals $\mathcal{O}_0^{\min}$. Hence $\mathcal{O}_0^F = \mathcal{O}_0^{\min} = \mathcal{O}_0^{\max}$.

Lemma 2.28 (Endpoint behaviour at $x = 0$) *Let $\mu > 1/2$ and $\epsilon > 0$. Suppose $f \in L^2(0,\epsilon)$ satisfies, in the* *distributional sense,*

$$-f'' + \frac{\mu(\mu+1)}{x^2} f = h, \qquad h \in L^2(0,\epsilon). \tag{122}$$

Then $f \in H^2_{\mathrm{loc}}(0,\epsilon)$ *and* $f/x^2 \in L^2(0,\epsilon)$.

Step 1 (interior regularity). On every compact $[a,b] \subset (0,\epsilon)$, the coefficient $\mu(\mu+1)/x^2$ is smooth and bounded. Since $f \in L^2_{\mathrm{loc}}(0,\epsilon)$ and $-f'' + (\text{bounded})f = h \in L^2_{\mathrm{loc}}(0,\epsilon)$, standard ODE regularity [52], [53] gives $f \in H^2_{\mathrm{loc}}(0,\epsilon)$; in particular, Eq. (122) holds a.e. on $(0,\epsilon)$.

Step 2 (Frobenius decomposition). By the theory of regular singular points [54, Ch. XV], [50, Ch. 4], on $(0,\epsilon)$ the function f decomposes as

$$f = c_1\phi_1 + c_2\phi_2 + f_p, \tag{123}$$

where $\phi_1(x) = x^{\mu+1}$ and $\phi_2(x) = x^{-\mu}$ are the two Frobenius solutions of the homogeneous equation $-y'' + \mu(\mu+1)y/x^2 = 0$, and f_p is a particular solution.

Step 3 ($c_2 = 0$). Since $f \in L^2(0,\epsilon)$ and $\phi_2(x) = x^{-\mu}$ with $\mu > 1/2$ satisfies

$$\int_0^\epsilon |\phi_2(x)|^2\, dx = \int_0^\epsilon x^{-2\mu}\, dx = +\infty,$$

the term $c_2\phi_2$ cannot appear in f. Hence $c_2 = 0$.

Step 4 (Green's function representation). Choose the particular solution of Eq. (122) given by the Green's function

$$f_p(x) \int_0^\epsilon G(x,y)h(y)\, dy, \quad G(x,y)$$
$$\frac{1}{2\mu+1}\left\{x^{\mu+1}y^{-\mu}, x<y, y^{\mu+1}x^{-\mu}, x>y.\right. \tag{124}$$

This G satisfies $(-d^2/dx^2 + \mu(\mu+1)/x^2)G(\cdot,y) = \delta_y$ distributionally, and $G(0,y) = 0$, so f_p is well-defined.

Step 5 (Schur test for the weighted kernel). Set $K(x,y)G(x,y)/x^2$. Then

$$K(x,y) = \frac{1}{2\mu+1}\begin{cases} x^{\mu-1}y^{-\mu}, & x<y, \\ y^{\mu+1}x^{-\mu-2}, & x>y. \end{cases} \tag{125}$$

We apply the Schur test with weight $p(x)x^{-1/2}$.

Row sum. For $x \in (0,\epsilon)$:

$$\begin{aligned} S_r(x) &= x^{1/2}\int_0^\epsilon K(x,y)\, y^{-1/2}dy \\ &= \frac{x^{1/2}}{2\mu+1}\left[x^{-\mu-2}\int_0^x y^{\mu+1/2}dy + x^{\mu-1}\int_x^\epsilon y^{-\mu-1/2}dy\right] \\ &= \frac{1}{2\mu+1}\left[\frac{1}{\mu+3/2} + \frac{1-(x/\epsilon)^{\mu-1/2}}{\mu-1/2}\right] \\ &\le \frac{1}{2\mu+1}\left[\frac{1}{\mu+3/2} + \frac{1}{\mu-1/2}\right] =: C_1 < \infty, \end{aligned} \tag{126}$$

since $\mu - 1/2 > 0$ implies $(x/\epsilon)^{\mu-1/2} \in (0,1)$ for $x \in (0,\epsilon)$.

Column sum. For $y \in (0,\epsilon)$:

$$\begin{aligned} S_c(y) &= y^{1/2}\int_0^\epsilon K(x,y)\, x^{-1/2}dx \\ &= \frac{y^{1/2}}{2\mu+1}\left[y^{-\mu}\int_0^y x^{\mu-3/2}dx + y^{\mu+1}\int_y^\epsilon x^{-\mu-5/2}dx\right] \\ &= \frac{1}{2\mu+1}\left[\frac{1}{\mu-1/2} + \frac{1-(y/\epsilon)^{\mu+3/2}}{\mu+3/2}\right] \\ &\le \frac{1}{2\mu+1}\left[\frac{1}{\mu-1/2} + \frac{1}{\mu+3/2}\right] = C_1 < \infty. \end{aligned} \tag{127}$$

Conclusion of Step 5. By the Schur test, the operator $T_K g(x) \int_0^\epsilon K(x,y)g(y)dy$ is bounded on $L^2(0,\epsilon)$ with $\|T_K\| \le C_1 < \infty$. Hence $f_p/x^2 = T_K h \in L^2(0,\epsilon)$.

Step 6 (assembly). From Steps 3 and 5:

$$\frac{f}{x^2} = \frac{c_1\phi_1 + f_p}{x^2} = c_1 x^{\mu-1} + \frac{f_p}{x^2}.$$

Now $\int_0^\epsilon |x^{\mu-1}|^2 dx = \int_0^\epsilon x^{2\mu-2}dx < \infty$ because $2\mu - 2 > -1 \iff \mu > 1/2$, so $c_1 x^{\mu-1} \in L^2(0,\epsilon)$. Combined with $f_p/x^2 \in L^2(0,\epsilon)$, we get $f/x^2 \in L^2(0,\epsilon)$.

Lemma 2.29 (Endpoint behaviour at $x = \infty$ via Agmon estimates]
Let $\omega > 0$ and $M > 0$. Let $V \in C^\infty(M,\infty)$ be real-valued with

$$\begin{aligned} &V(x) = \omega^2x^2 + O(x^{-2}) \quad \text{and} \\ &V'(x) = 2\omega^2 x + O(x^{-3}) \quad (x\to\infty). \end{aligned} \tag{128}$$

Suppose $f \in L^2(M,\infty)$ satisfies $-f'' + Vf = h \in L^2(M,\infty)$ distributionally. Then there exists $C > 0$ and $N > 0$ such that

$$|f(x)| \le Cx^N e^{-\omega x^2/2}, \quad x \ge M. \tag{129}$$

In particular, $x^2 f \in L^2(M,\infty)$.

This is a standard Agmon-type estimate for Schrödinger operators with confining potentials. We refer to [55, Theorem 3.1] and [56, Chapter 3] for the complete proof.

Sketch of the argument. Define the Agmon metric $\rho(x,y)| \int_x^y \sqrt{V(t)}dt|$ and set $H_M - d^2/dx^2 + V$ on $L^2(M,\infty)$ with Dirichlet condition at M. For any $\theta \in (0,1)$, there exists $C_\theta < \infty$ such that for every $g \in L^2(M,\infty)$,

$$\int_M^\infty e^{2\theta\rho(M,x)}|(H_M - E)^{-1}g(x)|^2 dx \le C_\theta\|g\|^2,$$

where $E < \inf\sigma(H_M)$. This is the classical Combes–Thomas / Agmon estimate. Writing $f = (H_M - E)^{-1}(h - Ef)$ and using interior H^2-regularity to pass from the L^2-weighted bound to pointwise bounds, we obtain Eq. (129) with $N = 1$ (or larger, depending on the subleading behaviour of V).
Since $\rho(M,x) = \omega x^2/2 + O(1)$ by Eq. (128), the exponential weight is $e^{\theta\omega x^2}$ at leading order, giving Eq. (129). The claim $x^2 f \in L^2(M,\infty)$ follows immediately:

$$\int_M^\infty x^4|f(x)|^2 dx \le C^2\int_M^\infty x^{4+2N}e^{-\omega x^2}dx < \infty.$$

Theorem 2.30 (Domain description) *For every $\mu > 1/2$ and $\omega > 0$, the domain $\mathcal{D}(\mathcal{O}_0)$ of Definition 2.2 coincides with the domains of the minimal, maximal, and Friedrichs operators:*

$$\boxed{\mathcal{O}_0^{\min} = \mathcal{O}_0^{\max} = \mathcal{O}_0^F = \mathcal{O}_0 \quad \text{on} \quad \mathcal{D}(\mathcal{O}_0),} \tag{130}$$

and all four operators have the same domain.

Step 1 (operator identity among $\mathcal{O}_0^{\min}$, $\mathcal{O}_0^{\max}$, $\mathcal{O}_0^F$). By Lemma 2.17, the quadratic form q_0 is densely defined, closed, and lower-semibounded. By Remark 2.18, q_0 admits a Friedrichs extension $\mathcal{O}_0^F$ with $\mathcal{O}_0^F \supseteq A_0$ and $\mathcal{D}(\mathcal{O}_0^F) \subseteq \mathcal{D}(q_0)$. By Lemma 2.25, both endpoints are of limit-point type, so by Corollary 2.26, $\mathcal{O}_0^{\min} = \mathcal{O}_0^{\max}$ is the unique self-adjoint extension of A_0. Applying this uniqueness to $\mathcal{O}_0^F$ gives $\mathcal{O}_0^F = \mathcal{O}_0^{\min}$, hence

$$\mathcal{O}_0^{\min} = \mathcal{O}_0^{\max} = \mathcal{O}_0^F. \tag{131}$$

Step 2 (inclusion $\mathcal{D}(\mathcal{O}_0) \subseteq \mathcal{D}(\mathcal{O}_0^{\max})$). For $f \in \mathcal{D}(\mathcal{O}_0)$: $f \in H^2$, $f/x^2 \in L^2$, $x^2 f \in L^2$. Hence

$$\mathcal{O}_0 f = -f'' + \mu(\mu+1)f/x^2 + \omega^2 x^2 f \in L^2(0,\infty)$$

(as a sum of three L^2-functions). By Definition 2.23, $f \in \mathcal{D}(\mathcal{O}_0^{\max})$.

Step 3 (reverse inclusion $\mathcal{D}(\mathcal{O}_0^{\max}) \subseteq \mathcal{D}(\mathcal{O}_0)$). Let $f \in \mathcal{D}(\mathcal{O}_0^{\max})$, so $f \in L^2(0,\infty)$ and $g\mathcal{O}_0 f \in L^2(0,\infty)$ distributionally.

(3a) Interior regularity. On every compact $[1, M] \subset (0,\infty)$, the potential V is smooth and bounded, so standard ODE regularity [52], [53] gives $f \in H^2(1, M)$.

(3b) $f' \in L^2$ via Friedrichs. By Eq. (131), $\mathcal{O}_0^{\max} = \mathcal{O}_0^F$. By the Friedrichs construction (Remark 2.18), $\mathcal{D}(\mathcal{O}_0^F) \subseteq \mathcal{D}(q_0)$ with $q_0(f) = \langle \mathcal{O}_0^F f, f\rangle$. Hence for $f \in \mathcal{D}(\mathcal{O}_0^{\max}) = \mathcal{D}(\mathcal{O}_0^F)$:

$$q_0(f) = \|f'\|^2 + \mu(\mu+1)\left\|\frac{f}{x}\right\|^2 + \omega^2\|xf\|^2 < \infty. \tag{132}$$

In particular, $f' \in L^2(0,\infty)$, $f/x \in L^2(0,\infty)$, $xf \in L^2(0,\infty)$.

(3c) Endpoint $x = 0$. On $(0,1)$, set $h_0 g - \omega^2 x^2 f$. Since $g \in L^2(0,1)$ and $|x^2 f| \le |f| \in L^2(0,1)$, we have $h_0 \in L^2(0,1)$. The equation $-f'' + \mu(\mu+1)f/x^2 = h_0$ holds distributionally, so Lemma 2.28 applies and yields $f/x^2 \in L^2(0,1)$.

(3d) Endpoint $x = \infty$. On (M,∞) with $M \ge 1$, the potential $V(x) = \omega^2 x^2 + O(x^{-2})$ satisfies Eq. (128). Lemma 2.29 applies with $h = g \in L^2(M,\infty)$ and gives $x^2 f \in L^2(M,\infty)$.

(3e) $f'' \in L^2$. From the equation $f'' = \mu(\mu+1)f/x^2 + \omega^2 x^2 f - g$, each term on the right is in $L^2(0,\infty)$: the first by (3c), the second by (3d), the third by hypothesis. Hence $f'' \in L^2(0,\infty)$.

(3f) Assembly. From (3b): $f' \in L^2$. From (3e): $f'' \in L^2$. Together with $f \in L^2$: $f \in H^2(0,\infty)$. Combined with $f/x^2 \in L^2$ (3c) and $x^2 f \in L^2$ (3d): $f \in \mathcal{D}(\mathcal{O}_0)$.

Step 4 (equality of domains and self-adjointness of $\mathcal{O}_0$). Steps 2 and 3 give $\mathcal{D}(\mathcal{O}_0) = \mathcal{D}(\mathcal{O}_0^{\max})$. Since $\mathcal{O}_0$ and $\mathcal{O}_0^{\max}$ act by the same differential expression on this common domain, we have $\mathcal{O}_0 = \mathcal{O}_0^{\max}$. By Step 1 and Corollary 2.26, $\mathcal{O}_0^{\max}$ is self-adjoint, hence so is $\mathcal{O}_0$. Combining with Eq. (131):

$$\mathcal{O}_0 = \mathcal{O}_0^{\min} = \mathcal{O}_0^{\max} = \mathcal{O}_0^F,$$

all with domain $\mathcal{D}(\mathcal{O}_0)$.

Remark 2.31 (Role of the analytic tools in the identification]

Two boundary lemmas enter the identification Eq. (130) directly:

- Lemma 2.28 (at $x = 0$): the Frobenius decomposition combined with the Schur test on the weighted Green's function G/x^2 gives $f/x^2 \in L^2(0,\epsilon)$ for every $f \in \mathcal{D}(\mathcal{O}_0^{\max})$; this is used in Step 3c of Theorem 2.30.
- Lemma 2.29 (at $x = \infty$): Agmon estimates give exponential decay $|f(x)| \le Cx^N e^{-\omega x^2/2}$, hence $x^2 f \in L^2(M,\infty)$; this is used in Step 3d of Theorem 2.30.

Additionally, the limit-point property of Lemma 2.25 (which uses the indicial equation at $x = 0$ and WKB at $x = \infty$) enters *indirectly* through the operator identity Eq. (131) in Step 1 of Theorem 2.30, via Corollary 2.26.

What is not *used.* Neither the Weyl–Titchmarsh theorem (invoked in Theorem 2.13, part (1), only to record self-adjointness of $\mathcal{O}_0^{\max}$) nor the spectral expansion of $\mathcal{O}_0$ enters the proof of Eq. (130). The identification $\mathcal{D}(\mathcal{O}_0) = \mathcal{D}(\mathcal{O}_0^{\max})$ is established independently of the spectral analysis of Theorem 2.13: its proof uses only Lemmas 2.28, 2.29, 2.25, the classical Friedrichs construction (Remark 2.18), and interior ODE regularity.

Role of boundary analysis. Boundary analysis at $x = 0$ enters the paper in three places: (i) in Remark 2.4, to justify the sharpness of the threshold $\mu > 1/2$ for the domain condition $f/x^2 \in L^2$; (ii) in Lemma 2.25, to establish the limit-point property underlying Corollary 2.26; and (iii) in Lemma 2.28, where the Frobenius decomposition plus Schur test is used in Step 3c of Theorem 2.30. Of these, only (iii) enters the identification argument *directly* (Step 3c); (ii) enters it *indirectly* through Corollary 2.26 in Step 1. Item (i) is completely external to the identification.

Remark 2.32 (Role of limit-point in uniqueness, and borderline cases]

Role of limit-point. The limit-point nature at both endpoints (Lemma 2.25) is the crucial hypothesis for Corollary 2.26. If the endpoint $x = 0$ were of limit-circle type (as in the case $\mu < 1/2$), there would be multiple self-adjoint extensions, and the domain $\mathcal{D}(\mathcal{O}_0)$ would not be unique, requiring an additional boundary condition at $x = 0$. The condition $\mu > 1/2$ excludes this possibility.

Case $\mu > 1/2$ (studied here). Both endpoints are of limit-point type. The self-adjoint extension is unique. $\mathcal{D}(\mathcal{O}_0)$ is naturally defined.

Case $\mu < 1/2$. The endpoint $x = 0$ is of limit-circle type ($2\mu < 1$, $\int_0^\epsilon x^{-2\mu}\,dx < \infty$). There are multiple self-adjoint extensions, requiring an additional boundary condition at $x = 0$. Outside the scope of this paper.

Borderline case $\mu = 1/2$. The endpoint $x = 0$ is still of limit-point type (since $2\mu = 1$, and $\int_0^\epsilon x^{-2\mu}\,dx = \int_0^\epsilon x^{-1}\,dx = \infty$ logarithmically). No additional boundary condition is needed to select a self-adjoint extension. However, the obstruction appears in the domain condition: the eigenfunctions satisfy $\psi_n(x)/x^2 \sim C_n x^{-1/2}$, and $\int_0^\epsilon x^{-1}\,dx = \infty$, so $\psi_n \notin \mathcal{D}(\mathcal{O}_0)$ by definition Eq. (20). The correct remedy is to relax the domain condition to a wider space (e.g. $\mathcal{D}(\mathcal{O}_0^{\max})$, which contains the eigenfunctions because $\mathcal{O}_0\psi_n = A_n\psi_n \in L^2$). This differs essentially from the case $\mu < 1/2$: there, the problem is uniqueness of the extension; here, the problem is selecting an appropriate domain.

Essential difference between the two borderline cases. At $\mu = 1/2$, the endpoint $x = 0$ is still of limit-point type, so uniqueness of the self-adjoint extension is guaranteed; the obstruction is purely a domain issue. At $\mu < 1/2$, the endpoint $x = 0$ is of limit-circle type, so uniqueness fails and an extra boundary condition is required. This distinction is fundamental.

3. RELATIVE BOUNDEDNESS AND LOGARITHMIC MATRIX ELEMENTS

In this section we prove that the two perturbations $\mathcal{O}_0^\sigma$ and $\log x$ are relatively bounded with respect to the base operator $\mathcal{O}_0$, with relative bound zero. This is a prerequisite for applying the Kato–Rellich theorem to the full operator $\mathcal{H}(\lambda,\gamma)$. We then derive the closed-form expression for the logarithmic matrix elements $V_{mn}^{\log}$, prove a uniform bound on them, establish a precise regular expansion, and derive the complete second-order expansion of the eigenvalues. All proofs are self-contained, and all constants are explicit.

3.1. Relative Boundedness of the Perturbations

3.1.1. The Fractional Power

Theorem 3.1 (Relative boundedness of $\mathcal{O}_0^\sigma$) *Let $0 < \sigma < 1$. For every $\epsilon > 0$ there exists a constant $C_\epsilon = C_{\epsilon,\sigma,\omega,\mu} > 0$ (independent of n) such that*

$$\|\mathcal{O}_0^\sigma f\| \le \epsilon\|\mathcal{O}_0 f\| + C_\epsilon\|f\|, \qquad \forall f \in \mathcal{D}(\mathcal{O}_0). \tag{133}$$

Explicit form of the constant:

$$C_\epsilon^2 \max\left(0, \sup_{t\ge A_0}(t^{2\sigma} - \epsilon^2 t^2)\right), \qquad A_0\omega(2\mu+3). \tag{134}$$

Since $\mathcal{D}(\mathcal{O}_0) \subseteq \mathcal{D}(\mathcal{O}_0^\sigma)$ (from Definition 2.6: $A_n \ge A_0 > 0$ and $\sigma < 1$, so $A_n^{2\sigma} \le A_n^{2(\sigma-1)}A_n^2$), we write $f = \sum_{n\ge0} c_n\psi_n$. Since $\mathcal{O}_0^\sigma$ is diagonal in this basis ($\mathcal{O}_0^\sigma\psi_n = A_n^\sigma\psi_n$), we obtain

$$\|\mathcal{O}_0^\sigma f\|^2 = \sum_n A_n^{2\sigma}|c_n|^2, \quad \|\mathcal{O}_0 f\|^2 = \sum_n A_n^2|c_n|^2, \quad \|f\|^2 = \sum_n |c_n|^2.$$

Step 1 (pointwise inequality in A_n). We prove $A_n^{2\sigma} \le \epsilon^2 A_n^2 + C_\epsilon^2$ for every n. The function $h(t)t^{2\sigma} - \epsilon^2 t^2$ on $[A_0,\infty)$ is continuous, and $h(t) \to -\infty$ as $t \to \infty$ (since $2\sigma < 2$). Hence h is bounded above on $[A_0,\infty)$, and its supremum is finite. By definition Eq. (134), $C_\epsilon^2 \ge h(t)$ for every $t \ge A_0$, i.e., $t^{2\sigma} \le \epsilon^2 t^2 + C_\epsilon^2$. Applying this to $t = A_n \ge A_0$ yields the pointwise inequality.

Step 2 (summation). Multiplying by $|c_n|^2$ and summing over n:

$$\|\mathcal{O}_0^\sigma f\|^2 = \sum_n A_n^{2\sigma}|c_n|^2 \le \sum_n(\epsilon^2 A_n^2 + C_\epsilon^2)|c_n|^2 = \epsilon^2\|\mathcal{O}_0 f\|^2 + C_\epsilon^2\|f\|^2.$$

Step 3 (square root). Using $\sqrt{a^2+b^2} \le a + b$ for $a, b \ge 0$:

$$\|\mathcal{O}_0^\sigma f\| \le \epsilon\|\mathcal{O}_0 f\| + C_\epsilon\|f\|.$$

Remark 3.2 (Relative bound zero) *Since $\epsilon > 0$ is arbitrary, the relative bound of $\mathcal{O}_0^\sigma$ with respect to $\mathcal{O}_0$ is zero. Precisely: for every $a > 0$ there exists $b < \infty$ such that $\|\mathcal{O}_0^\sigma f\| \le a\|\mathcal{O}_0 f\| + b\|f\|$. Thus $\mathcal{O}_0^\sigma$ is a "small perturbation" in the Kato sense, and the standard spectral theorems (Kato–Rellich [1], Weyl [4]) are applicable.*

3.1.2. Operator Boundedness of $\log x$

Theorem 3.3 (Operator boundedness of $\log x$) *For every $\mu > 1/2$ and every $\epsilon > 0$ there exists a constant $C'_\epsilon > 0$ (independent of f) such that*

$$\|\log x \cdot f\| \le \epsilon\|\mathcal{O}_0 f\| + C'_\epsilon\|f\|, \qquad \forall f \in \mathcal{D}(\mathcal{O}_0). \tag{135}$$

We split $(0,\infty)$ into three regions and treat each separately.

Step 1 (small x region). Define $g(x)\sqrt{x}\log(1/x)$ on $(0,1]$. Its derivative is $g'(x) = -(\log x + 2)/(2\sqrt{x})$. Setting $g'(x) = 0$: $x = e^{-2}$, with $g(e^{-2}) = 2/e$. Hence g has a unique maximum on $(0,1]$ at $x = e^{-2}$ with value $2/e$. For every $\epsilon_1 > 0$, choose $\delta \in (0, e^{-2}]$ such that $\sqrt{\delta}\log(1/\delta) \le \epsilon_1$ (possible since $\sqrt{\delta}\log(1/\delta) \to 0$ as $\delta \to 0^+$). On $(0,\delta]$, $|\log x| \le \epsilon_1 x^{-1/2}$. Hence

$$\int_0^\delta (\log x)^2|f|^2\,dx \le \epsilon_1^2\int_0^\delta x^{-1}|f|^2\,dx \le \frac{\epsilon_1^2}{\mu(\mu+1)}q_0(f),$$

using $x^{-1} \le x^{-2}$ on $(0, \delta]$ with $\delta \le e^{-2} < 1$.
Step 2 (intermediate region). On $[\delta, M]$, $\log x$ is bounded. Define $L_{\delta,M} \max(|\log\delta|, \log M)$. Then $\int_\delta^M (\log x)^2 |f|^2\, dx \le L_{\delta,M}^2 \|f\|^2$.
Step 3 (large x region). For every $\epsilon_2 > 0$, choose $M \ge 1$ large enough so that $\log x \le \epsilon_2 x$ for $x \ge M$ (possible since $\log x/x \to 0$ as $x \to \infty$). Hence $\int_M^\infty (\log x)^2 |f|^2\, dx \le \epsilon_2^2 \int_M^\infty x^2 |f|^2\, dx \le (\epsilon_2^2/\omega^2) q_0(f)$.
Step 4 (combination). Adding the three estimates:

$$\|\log x \cdot f\|^2 \le \left(\frac{\epsilon_1^2}{\mu(\mu+1)} + \frac{\epsilon_2^2}{\omega^2} \right) q_0(f) + L_{\delta,M}^2 \|f\|^2.$$

Step 5 (choice of ϵ_1, ϵ_2). Choose $\epsilon_1 (\epsilon/2)\sqrt{\mu(\mu+1)}$ and $\epsilon_2 (\epsilon/2)\omega$. Then $\epsilon_1^2/[\mu(\mu+1)] + \epsilon_2^2/\omega^2 = \epsilon^2/4 + \epsilon^2/4 = \epsilon^2/2$, so

$$\|\log x \cdot f\|^2 \le \frac{\epsilon^2}{2} q_0(f) + L_{\delta,M}^2 \|f\|^2.$$

Step 6 (interpolation inequality). From Lemma 2.20, $q_0(f) \le \|\mathcal{O}_0 f\| \|f\|$. Using $2ab \le a^2 + b^2$:

$$\|\log x \cdot f\|^2 \le \frac{\epsilon^2}{4} \|\mathcal{O}_0 f\|^2 + \left(L_{\delta,M}^2 + \frac{\epsilon^2}{4} \right) \|f\|^2.$$

Step 7 (final form). Set $\epsilon' \epsilon/2$ and $C'_{\epsilon'} \sqrt{L_{\delta,M}^2 + \epsilon^2/4}$. Then $\|\log x \cdot f\| \le \epsilon' \|\mathcal{O}_0 f\| + C'_{\epsilon'} \|f\|$. Since $\epsilon > 0$ is arbitrary, so is ϵ'.

Remark 3.4 (Role of the three-region decomposition)
The decomposition is essential. If we used only the bound $|\log x| \le C_0 x^{-1/2}$ on $(0, 1]$ with $C_0 = 2/e$ (the maximum of $\sqrt{x}\log(1/x)$), we would obtain $C_0^2/(\mu(\mu+1)) = 4/(e^2\mu(\mu+1))$, which for $\mu = 1/2$ gives $16/(3e^2) \approx 0.72$, not small. The three-region strategy exploits $\sqrt{x}\log(1/x) \to 0$ as $x \to 0^+$ to obtain arbitrary smallness in the small-x region, separately from the large-x region, which is treated differently. This is essential for obtaining arbitrary ϵ, and hence a zero relative bound.

3.2. Self-Adjointness of the Full Operator

Proposition 3.5 (Self-adjointness and discrete spectrum) *Let $\mu > 1/2$, $0 < \sigma < 1$, $|\lambda| \le \Lambda_{\text{gap}}$, and $|\gamma| \le \gamma_0^{\text{FP}}$. Then the full operator*

$$\mathcal{H}(\lambda, \gamma) = \mathcal{O}_0 + \lambda \mathcal{O}_0^\sigma + \gamma \log x \tag{136}$$

on the domain $\mathcal{D}(\mathcal{O}_0)$ is self-adjoint, bounded from below, and has purely discrete spectrum.

We apply the Kato–Rellich theorem. Set $B \lambda \mathcal{O}_0^\sigma + \gamma \log x$, so that $\mathcal{H} = \mathcal{O}_0 + B$.
Step 1 (domain inclusion). For $f \in \mathcal{D}(\mathcal{O}_0)$: $\mathcal{O}_0^\sigma f \in L^2$ by Theorem 3.1; $\log x \cdot f \in L^2$ by Theorem 3.3. Hence $Bf \in L^2$, and the domain of B contains $\mathcal{D}(\mathcal{O}_0)$.
Step 2 (symmetry). $\mathcal{O}_0^\sigma$ is self-adjoint on $\mathcal{D}(\mathcal{O}_0^\sigma) \supseteq \mathcal{D}(\mathcal{O}_0)$. $\log x$ is self-adjoint on $\mathcal{D}(\log x) \supseteq \mathcal{D}(\mathcal{O}_0)$. Hence B is symmetric on $\mathcal{D}(\mathcal{O}_0)$.
Step 3 (relative bound). From Theorems 3.1 and 3.3:

$$\|Bf\| \le (|\lambda|\epsilon_1 + |\gamma|\epsilon_2)\|\mathcal{O}_0 f\| + (|\lambda| C_{\epsilon_1}^{(1,\text{op})} + |\gamma| C_{\epsilon_2}^{(2,\text{op})})\|f\|,$$

where $C_{\epsilon_1}^{(1,\text{op})} = C_{\epsilon_1}$ is the constant from Theorem 3.1 and $C_{\epsilon_2}^{(2,\text{op})} = C'_{\epsilon_2}$ is the constant from Theorem 3.3. Choose $\epsilon_1 1/(4\max(|\lambda|, 1))$ and $\epsilon_2 1/(4\max(|\gamma|, 1))$, giving $|\lambda|\epsilon_1 + |\gamma|\epsilon_2 \le 1/4 + 1/4 = 1/2$. Thus

$$\|Bf\| \le \frac{1}{2}\|\mathcal{O}_0 f\| + b\|f\|,$$

where $b |\lambda| C_{\epsilon_1}^{(1,\text{op})} + |\gamma| C_{\epsilon_2}^{(2,\text{op})} < \infty$ depends on $\lambda, \gamma, \mu, \omega, \sigma$ but not on f.
Step 4 (Kato–Rellich). By the Kato–Rellich theorem: since $\mathcal{O}_0$ is self-adjoint on $\mathcal{D}(\mathcal{O}_0)$ (Theorem 2.13, part (1)), B is symmetric on $\mathcal{D}(B) \supseteq \mathcal{D}(\mathcal{O}_0)$, and the relative bound is $1/2 < 1$, the operator $\mathcal{H} = \mathcal{O}_0 + B$ is self-adjoint on $\mathcal{D}(\mathcal{O}_0)$ and bounded from below.
Step 5 (discrete spectrum). We prove $B(\mathcal{O}_0 + i)^{-1}$ is compact, so by Weyl's theorem [4], $\sigma_{\text{ess}}(\mathcal{H}) = \sigma_{\text{ess}}(\mathcal{O}_0) = \emptyset$.
(5a) $\mathcal{O}_0^\sigma(\mathcal{O}_0 + i)^{-1}$ is compact. In the eigenbasis, this operator is diagonal: $\mathcal{O}_0^\sigma(\mathcal{O}_0 + i)^{-1}\psi_n = [A_n^\sigma/(A_n + i)]\psi_n$. The singular values are

$$\sigma_n \frac{A_n^\sigma}{\sqrt{A_n^2 + 1}} \le A_n^{\sigma - 1} \to 0 \quad (n \to \infty),$$

since $\sigma - 1 < 0$ and $A_n \to \infty$. Hence the operator is compact.
(5b) $\log x(\mathcal{O}_0 + i)^{-1}$ is compact. We use finite-rank approximation. For $N \ge 0$, let P_N be the orthogonal projection onto $\text{span}\{\psi_0, \ldots, \psi_N\}$. Define

$$T \log x(\mathcal{O}_0 + i)^{-1}, \qquad T_N T P_N = \log x(\mathcal{O}_0 + i)^{-1} P_N.$$

The operator T_N has finite rank $\le N + 1$, since its range is contained in $\text{span}\{\log x \cdot \psi_0, \ldots, \log x \cdot \psi_N\}$. For each $m \le N$: $\log x \cdot \psi_m \in L^2$ since $\psi_m \in \mathcal{D}(\mathcal{O}_0)$ and Theorem 3.3 applies. Hence T_N is compact.
We show $\|T - T_N\| \to 0$ as $N \to \infty$. Let $g \in L^2$ with $\|g\| \le 1$, and set $f(\mathcal{O}_0 + i)^{-1}(I - P_N)g$. Then $f \in \mathcal{D}(\mathcal{O}_0)$ and $(I - P_N)$ projects onto the closure of $\text{span}\{\psi_{N+1}, \psi_{N+2}, \ldots\}$. Using $\mathcal{O}_0(\mathcal{O}_0 + i)^{-1} = I - i(\mathcal{O}_0 + i)^{-1}$:

$$\|f\| \le \frac{1}{A_{N+1}}, \qquad \|\mathcal{O}_0 f\| \le 1 + \frac{1}{A_{N+1}}.$$

By Theorem 3.3 with $\eta > 0$:

$$\|\log x \cdot f\| \le \eta\left(1 + \frac{1}{A_{N+1}}\right) + \frac{C'_\eta}{A_{N+1}} = \eta + \frac{\eta + C'_\eta}{A_{N+1}}.$$

Taking the supremum over $\|g\| \le 1$:

$$\|T - T_N\| \le \eta + \frac{\eta + C'_\eta}{A_{N+1}}.$$

For every $\delta > 0$, choose $\eta = \delta/2$ and N large enough so that $(\eta + C'_\eta)/A_{N+1} < \delta/2$ (possible since $A_{N+1} \to \infty$). Then $\|T - T_N\| < \delta$. Since $\delta > 0$ is arbitrary, T is the operator-norm limit of finite-rank operators, hence compact.

(5c) Conclusion. Since $\mathcal{O}_0^\sigma(\mathcal{O}_0 + i)^{-1}$ and $\log x(\mathcal{O}_0 + i)^{-1}$ are compact, and the space of compact operators is a linear subspace, the operator

$$B(\mathcal{O}_0 + i)^{-1} = \lambda \mathcal{O}_0^\sigma(\mathcal{O}_0 + i)^{-1} + \gamma \log x(\mathcal{O}_0 + i)^{-1}$$

is compact. By Weyl's theorem, $\sigma_{\text{ess}}(\mathcal{H}) = \sigma_{\text{ess}}(\mathcal{O}_0) = \emptyset$. Hence the spectrum is purely discrete.

Remark 3.6 (Explicit lower bound) *From the proof, $\mathcal{H}$ is bounded from below. We give an explicit lower bound. For every $f \in \mathcal{D}(\mathcal{O}_0)$, from Theorems 3.1 and 3.3 in quadratic-form form, with constants $C_{\epsilon_1}^{(1,\text{form})} \sup_{t \geq A_0}(t^\sigma - \epsilon_1 t)$ and $C_{\epsilon_2}^{(2,\text{form})} L_{\delta,M} + 1/2 + C'_{\epsilon_2}$, we have*

$$|\langle Bf, f\rangle| \leq \frac{1}{2} q_0(f) + C_{\text{lb}}\|f\|^2,$$

where $C_{\text{lb}} |\lambda| C_{\epsilon_1}^{(1,\text{form})} + |\gamma| C_{\epsilon_2}^{(2,\text{form})}$. Since $\langle \mathcal{H}f, f\rangle = q_0(f) + \langle Bf, f\rangle \geq q_0(f) - |\langle Bf, f\rangle| \geq \frac{1}{2}q_0(f) - C_{\text{lb}}\|f\|^2$, and using $q_0(f) \geq c_0\|f\|^2$ (Lemma 2.19) with $c_0 = 2\omega\sqrt{\mu(\mu+1)}$:

$$\langle \mathcal{H}f, f\rangle \geq \left(\frac{1}{2}c_0 - C_{\text{lb}}\right)\|f\|^2.$$

Hence $\mu_0 \frac{1}{2}c_0 - C_{\text{lb}} = \omega\sqrt{\mu(\mu+1)} - C_{\text{lb}}$ is an explicit lower bound. A positive value is guaranteed when $c_0 > 2C_{\text{lb}}$, i.e., for $|\lambda|, |\gamma|$ small enough.

3.3. Logarithmic Matrix Elements

We introduce the logarithmic matrix elements:

$$V_{mn}^{\log} \langle \psi_m, \log x\, \psi_n\rangle = \int_0^\infty \psi_m(x)\psi_n(x) \log x\, dx, \quad (137)$$

where $\{\psi_n\}_{n\geq 0}$ are the eigenfunctions defined in Theorem 2.13. These elements play a central role in the perturbation theory and in the fixed-point formulation of Section 4.3.

3.3.1. Closed-Form Formula

Lemma 3.7 (Closed form for $V_{mn}^{\log}$) *For $m \neq n$:*

$$\boxed{V_{mn}^{\log} = -\frac{1}{2|m-n|}\sqrt{\frac{\max(m,n)!\,\Gamma(\min(m,n)+\alpha+1)}{\min(m,n)!\,\Gamma(\max(m,n)+\alpha+1)}}.} \quad (138)$$

For $m = n$:

$$\boxed{V_{nn}^{\log} = \frac{1}{2}[\Psi(n+\alpha+1) - \log\omega],} \quad (139)$$

where $\Psi \Gamma'/\Gamma$ is the digamma function.

The proof uses the standard differentiation formula for generalized Laguerre polynomials (see, e.g., [32, Ch. V] and [37, eq. 18.9.19]):

$$\frac{\partial}{\partial\alpha} L_k^{(\alpha)}(y) = \sum_{j=0}^{k-1} \frac{1}{k-j} L_j^{(\alpha)}(y), \quad k \geq 1. \quad (140)$$

(For $k = 0$, the derivative vanishes.)

Step 1 (substitution). Set $y \omega x^2$. Then $x = (y/\omega)^{1/2}$, $dx = \frac{1}{2}\omega^{-1/2}y^{-1/2}dy$, and $\log x = \frac{1}{2}(\log y - \log\omega)$.

Step 2 (expression for $\psi_m\psi_n\, dx$). From Theorem 2.13:

$$\psi_m(x)\psi_n(x) = \mathcal{N}_m\mathcal{N}_n x^{2\alpha+1} e^{-\omega x^2} L_m^{(\alpha)}(\omega x^2) L_n^{(\alpha)}(\omega x^2).$$

Substituting $x^{2\alpha+1} = (y/\omega)^{\alpha+1/2}$ and $e^{-\omega x^2} = e^{-y}$:

$$\psi_m\psi_n\, dx = \mathcal{N}_m\mathcal{N}_n (y/\omega)^{\alpha+1/2} e^{-y} L_m^{(\alpha)}(y) L_n^{(\alpha)}(y) \cdot \frac{1}{2}\omega^{-1/2}y^{-1/2}\, dy.$$

Simplifying $(y/\omega)^{\alpha+1/2} \cdot \omega^{-1/2}y^{-1/2} = \omega^{-\alpha-1}y^\alpha$. Using $\mathcal{N}_m\mathcal{N}_n = 2\omega^{\alpha+1}\sqrt{m!n!/[\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)]}$:

$$\psi_m\psi_n\, dx = \sqrt{\frac{m!n!}{\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)}}\, y^\alpha e^{-y} L_m^{(\alpha)}(y) L_n^{(\alpha)}(y)\, dy.$$

Step 3 (final expression for $V_{mn}^{\log}$). Since $\log x = \frac{1}{2}(\log y - \log\omega)$:

$$V_{mn}^{\log} = \frac{1}{2}\sqrt{\frac{m!n!}{\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)}}[I_{mn} - \log\omega \cdot J_{mn}],$$

where $I_{mn} \int_0^\infty y^\alpha e^{-y} \log y\, L_m^{(\alpha)}(y) L_n^{(\alpha)}(y)\, dy$ and $J_{mn} \int_0^\infty y^\alpha e^{-y} L_m^{(\alpha)}(y) L_n^{(\alpha)}(y)\, dy = [\Gamma(m+\alpha+1)/m!]\delta_{mn}$.

Step 4 (computation of I_{mn} for $m \neq n$). We differentiate the orthogonality relation with respect to α:

$$0 = \frac{d}{d\alpha}\left[\int_0^\infty y^\alpha e^{-y} L_m^{(\alpha)}(y) L_n^{(\alpha)}(y)\, dy\right].$$

Applying the product rule:

$$0 = I_{mn} + \int_0^\infty y^\alpha e^{-y}(\partial_\alpha L_m^{(\alpha)}) L_n^{(\alpha)}\, dy + \int_0^\infty y^\alpha e^{-y} L_m^{(\alpha)}(\partial_\alpha L_n^{(\alpha)})\, dy.$$

Hence

$$I_{mn} = -\left[\int y^\alpha e^{-y}(\partial_\alpha L_m^{(\alpha)}) L_n^{(\alpha)}\, dy + \int y^\alpha e^{-y} L_m^{(\alpha)}(\partial_\alpha L_n^{(\alpha)})\, dy\right].$$

Case $m < n$ (i.e. $rn - m > 0$). From Eq. (140), $\partial_\alpha L_m^{(\alpha)} = \sum_{j=0}^{m-1}[1/(m-j)] L_j^{(\alpha)}(y)$, and each L_j has degree $j \leq m-1 < m < n$. By orthogonality, the first integral vanishes. For the second integral:

$$\int y^\alpha e^{-y} L_m^{(\alpha)}(\partial_\alpha L_n^{(\alpha)})\, dy = \sum_{j=0}^{n-1} \frac{1}{n-j} \int y^\alpha e^{-y} L_m^{(\alpha)} L_j^{(\alpha)}\, dy.$$

By orthogonality, only the term $j = m$ survives:

$$= \frac{1}{n-m} \int y^\alpha e^{-y} (L_m^{(\alpha)})^2 \, dy = \frac{1}{n-m} \cdot \frac{\Gamma(m+\alpha+1)}{m!}.$$

Hence $I_{mn} = -\Gamma(m+\alpha+1)/[(n-m)m!]$ for $m < n$.

Case $m > n$. By symmetry ($I_{mn} = I_{nm}$ since the $L_k^{(\alpha)}$ are real-valued): $I_{mn} = -\Gamma(n+\alpha+1)/[(m-n)n!]$.

Unified formula. Combining the two cases:

$$I_{mn} = -\frac{\Gamma(\min(m,n)+\alpha+1)}{|m-n| \min(m,n)!}, \qquad m \neq n. \tag{141}$$

Step 5 (conclusion for $m \neq n$). Substituting into the expression for $V_{mn}^{\log}$ with $\delta_{mn} = 0$:

$$V_{mn}^{\log} = \frac{1}{2} \sqrt{\frac{m!n!}{\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)}} \cdot \left(-\frac{\Gamma(\min(m,n)+\alpha+1)}{|m-n| \min(m,n)!} \right).$$

Simplifying the square-root factor:

$$\sqrt{\frac{m!n!}{\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)}} \cdot \frac{\Gamma(\min+\alpha+1)}{\min!} = \sqrt{\frac{\max! \cdot \min!}{\Gamma(\min+\alpha+1)\Gamma(\max+\alpha+1)}} \cdot \frac{\Gamma(\min+\alpha+1)}{\min!} = \sqrt{\frac{\max! \cdot \Gamma(\min+\alpha+1)}{\min! \cdot \Gamma(\max+\alpha+1)}}.$$

Hence

$$V_{mn}^{\log} = -\frac{1}{2|m-n|} \sqrt{\frac{\max(m,n)! \, \Gamma(\min(m,n)+\alpha+1)}{\min(m,n)! \, \Gamma(\max(m,n)+\alpha+1)}}.$$

Step 6 (computation of I_{nn}). We differentiate the orthogonality relation at $m = n$:

$$I_{nn} + 2 \int y^\alpha e^{-y} L_n^{(\alpha)} (\partial_\alpha L_n^{(\alpha)}) \, dy = \frac{\Gamma(n+\alpha+1)}{n!} \Psi(n+\alpha+1).$$

(We use $(d/d\alpha)\Gamma(z) = \Gamma(z)\Psi(z)$ with $z = n+\alpha+1$.) The second term vanishes: $\partial_\alpha L_n^{(\alpha)}$ has degree $< n$ (by Eq. (140)), so $L_n^{(\alpha)}$ is orthogonal to it. Hence

$$I_{nn} = \frac{\Gamma(n+\alpha+1)}{n!} \Psi(n+\alpha+1).$$

Step 7 (conclusion for $m = n$). Substituting into the expression for $V_{nn}^{\log}$ with $\delta_{nn} = 1$:

$$V_{nn}^{\log} = \frac{1}{2} \sqrt{\frac{n!n!}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+1)}} \cdot \frac{\Gamma(n+\alpha+1)\Psi(n+\alpha+1)}{n!} - \frac{1}{2} \log \omega$$
$$= \frac{1}{2} \cdot \frac{n!}{\Gamma(n+\alpha+1)} \cdot \frac{\Gamma(n+\alpha+1)\Psi(n+\alpha+1)}{n!} - \frac{1}{2} \log \omega$$
$$= \frac{1}{2} \Psi(n+\alpha+1) - \frac{1}{2} \log \omega.$$

Hence $V_{nn}^{\log} = \frac{1}{2}[\Psi(n+\alpha+1) - \log \omega]$.

Remark 3.8 (Special case $\alpha = 1$) *At the borderline value $\alpha = 1$ (i.e., $\mu = 1/2$, excluded from the main results by Remark 2.4), the closed-form formula* Eq. (138) *simplifies to a telescoping form:*

$$\boxed{|V_{mn}^{\log}| = \frac{1}{2|m-n|} \sqrt{\frac{\min(m,n)+1}{\max(m,n)+1}}.} \tag{142}$$

Remark 3.9 (Exact form of the diagonal difference) *From Lemma 3.7, for every $n \geq 0$ and $r \neq 0$ with $n + r \geq 0$:*

$$V_{n+r,n+r}^{\log} - V_{nn}^{\log} = \frac{1}{2}[\Psi(n+r+\alpha+1) - \Psi(n+\alpha+1)]. \tag{143}$$

In the special case $\alpha = 1$:

$$V_{n+r,n+r}^{\log} - V_{nn}^{\log} = \frac{1}{2}[H_{n+r+1} - H_{n+1}]. \tag{144}$$

3.3.2. Uniform Bound

Theorem 3.10 (Uniform bound) *For every $r \neq 0$ and $n \geq 0$ with $n + r \geq 0$:*

$$\boxed{|V_{n,n+r}^{\log}| \leq \frac{1}{2|r|}.} \tag{145}$$

In particular, $|V_{mn}^{\log}| \leq 1/(2|m-n|)$ for $m \neq n$.

From Lemma 3.7, for $m \neq n$:

$$|V_{mn}^{\log}|^2 = \frac{1}{4|m-n|^2} \cdot \frac{\max(m,n)!}{\min(m,n)!} \cdot \frac{\Gamma(\min(m,n)+\alpha+1)}{\Gamma(\max(m,n)+\alpha+1)}.$$

Case $r > 0$ (i.e. $n + r > n$). Set $m = n$ (lower) and $n' = n + r$ (upper). Then

$$|V_{n,n+r}^{\log}|^2 = \frac{1}{4r^2} \cdot \frac{(n+r)!}{n!} \cdot \frac{\Gamma(n+\alpha+1)}{\Gamma(n+r+\alpha+1)}.$$

We simplify: $(n+r)!/n! = \prod_{j=1}^r (n+j)$ and $\Gamma(n+r+\alpha+1)/\Gamma(n+\alpha+1) = \prod_{j=1}^r (n+j+\alpha)$. Hence

$$|V_{n,n+r}^{\log}|^2 = \frac{1}{4r^2} \prod_{j=1}^r \frac{n+j}{n+j+\alpha}.$$

Each factor $(n+j)/(n+j+\alpha) < 1$ (since $\alpha > 0$ and $n + j \geq 1$). Hence the product ≤ 1, so $|V_{n,n+r}^{\log}|^2 \leq 1/(4r^2)$, which gives $|V_{n,n+r}^{\log}| \leq 1/(2r)$.

Case $r < 0$. Set $s - r > 0$, so $n + r = n - s$ (with $n - s \geq 0$). By symmetry $V_{n,n-s}^{\log} = V_{n-s,n}^{\log}$ (since the matrix elements are real). Applying the previous case to the pair $(n-s, n)$ with $r' = s > 0$:

$$|V_{n,n-s}^{\log}| = |V_{n-s,n}^{\log}| \leq \frac{1}{2s} = \frac{1}{2|r|}.$$

Remark 3.11 (Interpretation) *The bound $|V_{n,n+r}^{\log}| \leq 1/(2|r|)$ means that the logarithmic matrix elements decay at rate $1/|r|$ away from the diagonal. This decay is "slow" (not exponential) and is the source of complexity in the subsequent analysis. Specifically, this slow decay is the source of the $\log n$ term in the second-order correction (see Theorem 3.28).*

3.3.3. Regular Expansion

Lemma 3.12 (Regular expansion) *For $1 \le r \le n$ and $n \ge n_\Delta^{(2)} \lceil 2\alpha \rceil$:*

$$|V_{n,n+r}^{\log}|^2 = \frac{1}{4r^2}\left(1+\frac{r}{n}\right)^{-\alpha}(1+O(n^{-1})), \qquad (146)$$

where $O(n^{-1})$ is uniform in $r \in [1,n]$.

Step 1 (setup). Set $P_r(n) \prod_{k=1}^{r}(n+k)/(n+k+\alpha)$, so that $|V_{n,n+r}^{\log}|^2 = P_r(n)/(4r^2)$.

Step 2 (logarithm expansion). For each $k \in [1,r]$:

$$\log\frac{n+k}{n+k+\alpha} = \log\left(1-\frac{\alpha}{n+k+\alpha}\right).$$

With $u\alpha/(n+k+\alpha) \le \alpha/(2\alpha+1) < 1/2$ for $n \ge 2\alpha$. Since $n_\Delta^{(2)} = \lceil 2\alpha \rceil \ge 2\alpha$, the hypothesis $n \ge n_\Delta^{(2)}$ implies $n \ge 2\alpha$. Thus the series converges uniformly in k and $n \ge n_\Delta^{(2)}$:

$$\log\frac{n+k}{n+k+\alpha} = -\frac{\alpha}{n+k+\alpha} - \frac{1}{2}\left(\frac{\alpha}{n+k+\alpha}\right)^2 + O\left(\frac{\alpha^3}{(n+k)^3}\right).$$

We simplify $\alpha/(n+k+\alpha)$:

$$\frac{\alpha}{n+k+\alpha} = \frac{\alpha}{n+k}\left(1-\frac{\alpha}{n+k}+O\left(\frac{\alpha^2}{(n+k)^2}\right)\right) = \frac{\alpha}{n+k} - \frac{\alpha^2}{(n+k)^2} + O\left(\frac{\alpha^3}{(n+k)^3}\right).$$

Hence $\log[(n+k)/(n+k+\alpha)] = -\alpha/(n+k) + \alpha^2/[2(n+k)^2] + O(\alpha^3/(n+k)^3)$.

Step 3 (summation). Define $S_r^{(j)} \sum_{k=1}^{r}(n+k)^{-j}$ for $j \ge 1$. Then

$$\log P_r(n) = \sum_{k=1}^{r}\log\frac{n+k}{n+k+\alpha} = -\alpha S_r^{(1)} + \frac{\alpha^2}{2}S_r^{(2)} + O(\alpha^3 S_r^{(3)}).$$

Step 4 (estimate of $S_r^{(j)}$ by integral comparison). The function $t \mapsto (n+t)^{-j}$ is decreasing on $[0,\infty)$ for $j \ge 1$. By the integral comparison theorem: for every $f : [1,\infty) \to [0,\infty)$ decreasing and $N \ge 1$,

$$\sum_{n \ge N} f(n) \le f(N) + \int_N^\infty f(t)\,dt.$$

Applying this to $f(t) = (n+t)^{-j}$ with $N = 1$:

$$\int_1^{r+1}\frac{dt}{(n+t)^j} \le S_r^{(j)} \le \frac{1}{(n+1)^j} + \int_1^r \frac{dt}{(n+t)^j}.$$

Case $j = 1$. We compute:

$$\int_1^{r+1}\frac{dt}{n+t} = \log\frac{n+r+1}{n+1}, \qquad \int_1^r \frac{dt}{n+t} = \log\frac{n+r}{n+1}.$$

Hence

$$\log\frac{n+r+1}{n+1} \le S_r^{(1)} \le \frac{1}{n+1} + \log\frac{n+r}{n+1}.$$

We verify precisely that $S_r^{(1)} = \log(1+r/n) + O(1/n)$. First,

$$\frac{n+r}{n+1} = \frac{n(1+r/n)}{n(1+1/n)} = \frac{1+r/n}{1+1/n},$$

so $\log[(n+r)/(n+1)] = \log(1+r/n) - \log(1+1/n) = \log(1+r/n) - 1/n + O(1/n^2)$. Therefore

$$|S_r^{(1)} - \log(1+r/n)| \le \frac{1}{n+1} + \frac{1}{n} + O(1/n^2) \le \frac{C}{n},$$

where the constant C is uniform in $r \in [1,n]$.

Case $j \ge 2$. We compute

$$\int_1^\infty (n+t)^{-j}\,dt = \frac{1}{(j-1)(n+1)^{j-1}} \le \frac{1}{(j-1)n^{j-1}}.$$

Thus $S_r^{(j)} \le (n+1)^{-j} + (j-1)^{-1}n^{-(j-1)} = O(n^{1-j})$, uniformly in r.

Step 5 (combination). We have $\log P_r(n) = -\alpha\log(1+r/n) + O(1/n)$, where we used:

- $\alpha \cdot O(1/n) = O(1/n)$ (since $\alpha = \mu + 1/2$ is constant);
- $\alpha^2/2 \cdot O(1/n) = O(1/n)$;
- $O(\alpha^3 \cdot O(n^{-2})) = O(n^{-2}) \subseteq O(1/n)$.

Step 6 (exponential). From $\log P_r(n) = -\alpha\log(1+r/n) + O(1/n)$:

$$P_r(n) = (1+r/n)^{-\alpha} \cdot e^{O(1/n)} = (1+r/n)^{-\alpha}(1+O(1/n)),$$

where $e^{O(1/n)} = 1 + O(1/n)$ (by $e^u = 1 + u + O(u^2)$ as $u \to 0$).

Remark 3.13 (Explicit form of the error term) *For $n \ge 2\alpha$ and $1 \le r \le n$, the expansion can be sharpened to*

$$\log P_r(n) = -\alpha\log\left(1+\frac{r}{n}\right) + \frac{\alpha(\alpha+1)}{2n}\cdot\frac{r}{n+r} + O\left(\frac{\alpha^3}{n^2}\right), \qquad (147)$$

with the constant in $O(\cdot)$ depending only on α and uniform in $r \in [1,n]$.

3.3.4. Matrix Element Difference δ_r^{mat}

Definition 3.14 (Matrix element difference) *For $1 \le r \le n$:*

$$\delta_r^{mat}|V_{n,n-r}^{\log}|^2 - |V_{n,n+r}^{\log}|^2. \qquad (148)$$

Lemma 3.15 (Estimate of δ_r^{mat}) *Let $n_\Delta^{(4)} \max(\lceil 4\alpha \rceil, \lceil 3(1+\alpha) \rceil)$. For $n \ge n_\Delta^{(4)}$ and $1 \le r \le \lfloor n/2 \rfloor$:*

$$\delta_r^{mat} = -\frac{\alpha}{4n^2}(1+O(\alpha r/n)). \qquad (149)$$

The constant in $O(\alpha r/n)$ depends on α only, not on n or r.

Step 1 (definition of F_k). Write $P_r^-/P_r^+ = \prod_{k=1}^{r} F_k$, where

$$F_k \frac{(n-k+1)(n+k+\alpha)}{(n+k)(n-k+1+\alpha)}.$$

Step 2 (expansion of F_k). We compute the numerator minus the denominator. Numerator:

$$(n-k+1)(n+k+\alpha) = (n+1-k)(n+k)+ (n-k+1)\alpha = n^2+n+k-k^2+n\alpha-k\alpha+\alpha.$$

Denominator:

$$(n+k)(n-k+1+\alpha) = (n+k)(n+1-k)+ (n+k)\alpha = n^2+n+k-k^2+n\alpha+k\alpha.$$

Difference (numerator minus denominator): $-\alpha(2k-1)$. Hence $F_k = 1+u_k$ where

$$u_k - \frac{\alpha(2k-1)}{(n+k)(n-k+1+\alpha)}.$$

Step 3 (estimate of $|u_k|$). For $1 \le k \le r \le \lfloor n/2 \rfloor$: $n-k+1+\alpha \ge n/2$. Hence

$$|u_k| \le \frac{\alpha(2k-1)}{(n+k)(n/2)} \le \frac{2\alpha(2k-1)}{n(n+k)} \le \frac{4\alpha k}{n^2} \le \frac{4\alpha r}{n^2} \le \frac{2\alpha}{n}.$$

Since $n_\Delta^{(4)} \ge \lceil 4\alpha \rceil \ge 4\alpha$, the hypothesis $n \ge n_\Delta^{(4)}$ implies $n \ge 4\alpha$, so $|u_k| \le 1/2$. (This justifies the subsequent expansion of $\log(1+u_k)$.)

Step 4 (expansion of u_k). Write

$$(n+k)(n-k+1+\alpha) = n^2+B_k, \quad B_k n(1+\alpha)+ k(1+\alpha)-k^2.$$

We note that

$$\frac{|B_k|}{n^2} \le \frac{n(1+\alpha)+k(1+\alpha)+k^2}{n^2} \le \frac{1+\alpha}{n} + \frac{(1+\alpha)r}{n^2} + \frac{r^2}{n^2}.$$

Using $r \le n/2$: $\frac{(1+\alpha)r}{n^2} \le \frac{1+\alpha}{2n}$ and $\frac{r^2}{n^2} \le \frac{1}{4}$, so

$$\frac{|B_k|}{n^2} \le \frac{1+\alpha}{n} + \frac{1+\alpha}{2n} + \frac{1}{4} = \frac{3(1+\alpha)}{2n} + \frac{1}{4}.$$

Since $n_\Delta^{(4)} \ge \lceil 3(1+\alpha) \rceil \ge 3(1+\alpha)$, the hypothesis $n \ge n_\Delta^{(4)}$ implies $n \ge 3(1+\alpha)$, so $\frac{3(1+\alpha)}{2n} \le \frac{1}{2}$ and $|B_k|/n^2 \le \frac{3}{4} < 1$. By the geometric expansion,

$$\frac{1}{1+B_k/n^2} = 1 - \frac{B_k}{n^2} + O(k^2/n^4).$$

Hence $u_k = -\alpha(2k-1)/n^2 \cdot (1-B_k/n^2+O(k^2/n^4))$.

Step 5 (summation). Set $S_r \sum_{k=1}^r u_k$. Then

$$S_r = -\frac{\alpha}{n^2}\sum_{k=1}^r (2k-1) + \frac{\alpha}{n^4}\sum_{k=1}^r (2k-1)B_k + O\left(\frac{\alpha}{n^4}\sum_{k=1}^r k^2(2k-1)\right).$$

We compute the sums:

$$\sum_{k=1}^r (2k-1) = r^2;$$

$$\sum_{k=1}^r k(2k-1) = \frac{r(r+1)(4r-1)}{6} = \frac{2r^3}{3} + O(r^2);$$

$$\sum_{k=1}^r k^2(2k-1) = \frac{r^4}{2} + O(r^3);$$

$$\sum_{k=1}^r (2k-1)B_k = n(1+\alpha)r^2 + (1+\alpha)\sum k(2k-1) - \sum k^2(2k-1).$$

Substituting:

$$S_r = -\frac{\alpha r^2}{n^2} + \frac{\alpha}{n^4}\Big[n(1+\alpha)r^2 + (1+\alpha)\frac{2r^3}{3} - \frac{r^4}{2} + O(r^2)+O(r^3)\Big] + O(\alpha r^4/n^4).$$

We simplify. The leading term is $-\alpha r^2/n^2$. The additional terms are: $\alpha(1+\alpha)r^2/n^3 = O(\alpha r^2/n^3)$; $2\alpha(1+\alpha)r^3/(3n^4) = O(\alpha r^3/n^4)$; $-\alpha r^4/(2n^4) = O(\alpha r^4/n^4)$; $O(\alpha r^4/n^4)$. For $r \le n/2$: $r^2/n^3 \le 1/(4n)$, $r^3/n^4 \le 1/(8n)$, $r^4/n^4 \le 1/16$. Hence

$$S_r = -\frac{\alpha r^2}{n^2}\Big(1+O(1/n)+O(r/n^2)+ O(r^2/n^2)+O(\alpha r/n).$$

We combine all error terms into $O(\alpha r/n)$ (since $\alpha > 0$, and $O(1/n) \le O(\alpha/n) \le O(\alpha r/n)$, $O(r/n^2) \le O(\alpha r/n)$, $O(r^2/n^2) \le O(\alpha r/n)$ for $r \le n/2$). Hence

$$S_r = -\frac{\alpha r^2}{n^2}(1+O(\alpha r/n)).$$

Step 6 (exponential). Since $|S_r| \le 2\alpha r/n \le \alpha$ (for $r \le n/2$ and $n \ge 4\alpha$), we use the elementary bound $|\exp(u)-1-u| \le |u|^2 e^{|u|}/2$, valid for all $u \in \mathbb{R}$. With $|S_r| \le \alpha$:

$$|\exp(S_r)-1-S_r| \le \frac{1}{2}|S_r|^2 e^{\alpha} = O_\alpha(S_r^2),$$

where the constant in $O_\alpha(\cdot)$ depends only on α (and not on n or r). Therefore

$$P_r^-/P_r^+ = \exp(S_r+O_\alpha(S_r^2)) = 1+S_r+O_\alpha(S_r^2) = 1-\frac{\alpha r^2}{n^2}(1+O_\alpha(\alpha r/n)).$$

Step 7 (difference).

$$P_r^- - P_r^+ = P_r^+ \cdot (P_r^-/P_r^+ - 1) = P_r^+ \cdot \left(-\frac{\alpha r^2}{n^2}\right)(1+O(\alpha r/n)).$$

From Lemma 3.12, $P_r^+ = (1+r/n)^{-\alpha}(1+O(1/n)) = 1+O(r/n)$. Hence

$$P_r^- - P_r^+ = -\frac{\alpha r^2}{n^2}(1+O(\alpha r/n)).$$

Step 8 (conclusion). $\delta_r^{\text{mat}} = (P_r^- - P_r^+)/(4r^2) = -\alpha/(4n^2)(1+O(\alpha r/n))$.

Remark 3.16 (Sign of δ_r^{mat}) *Since $\alpha > 0$ and $n > 0$, we have $\delta_r^{mat} < 0$ for small values of r/n. This means $|V_{n,n-r}^{\log}| < |V_{n,n+r}^{\log}|$. This asymmetry is the source of the* $\log n$ *term in the second-order correction (see Theorem 3.28).*

Lemma 3.17 (Exact form of δ_r^{mat}) *For every $\alpha > 1$, $n \geq 1$, and $1 \leq r \leq n$:*

$$\delta_r^{mat} = \frac{1}{4r^2}\left[\prod_{k=1}^{r}\frac{n-k+1}{n-k+1+\alpha} - \prod_{k=1}^{r}\frac{n+k}{n+k+\alpha}\right]. \quad (150)$$

Direct application of Lemma 3.7: $|V_{n,n-r}^{\log}|^2 = (1/(4r^2))\prod_{k=1}^{r}(n-k+1)/(n-k+1+\alpha)$ and $|V_{n,n+r}^{\log}|^2 = (1/(4r^2))\prod_{k=1}^{r}(n+k)/(n+k+\alpha)$.

Lemma 3.18 (Exact form at $\alpha = 1$) *At the borderline value $\alpha = 1$, for $1 \leq r \leq n$:*

$$\boxed{\delta_r^{mat} = -\frac{1}{4(n+1)(n+r+1)}.} \quad (151)$$

At $\alpha = 1$, both products telescope:

$$\prod_{k=1}^{r}\frac{n-k+1}{n-k+2} = \frac{n-r+1}{n+1}, \qquad \prod_{k=1}^{r}\frac{n+k}{n+k+1} = \frac{n+1}{n+r+1}.$$

The numerator of the difference is $(n-r+1)(n+r+1) - (n+1)^2 = (n+1)^2 - r^2 - (n+1)^2 = -r^2$. Therefore

$$\delta_r^{mat} = \frac{1}{4r^2}\cdot\frac{-r^2}{(n+1)(n+r+1)} = -\frac{1}{4(n+1)(n+r+1)}.$$

Remark 3.19 (Role of Lemma 3.18) *The case $\alpha = 1$ (equivalently, $\mu = 1/2$) is excluded from the main scope of the paper (see Remark 2.4). Lemma 3.18 is included for two reasons: (i) it illustrates the telescoping structure of δ_r^{mat} in the borderline case, providing intuition for the general $\alpha > 1$ case; and (ii) it confirms the general scaling $\delta_r^{mat} \sim -c_\alpha/n^2$ from Lemma 3.15 with $c_1 = 1/4$. No result in the main text depends on Lemma 3.18.*

3.3.5. Effective Energy Gap and the Quantity H_n

We first introduce the quantity H_n, which appears both in the effective gap and in the second-order expansion.

Definition 3.20 (The quantity H_n) *For $n \geq 1$:*

$$\boxed{H_n 1 + \lambda\sigma(4\omega n)^{\sigma-1}.} \quad (152)$$

We also set $H_n^* 1 + \lambda\sigma A_n^{\sigma-1} = H_n + O(1/n)$.

Remark 3.21 (Exclusion of $n = 0$) *We exclude $n = 0$ since $(4\omega n)^{\sigma-1} = 0^{-(1-\sigma)} = \infty$ (as $\sigma < 1$). This difference in behaviour at $n = 0$ is one reason why the case $n = 0$ must be treated separately in Section 4.4.*

Lemma 3.22 (Positivity of H_n) *Under the hypothesis $|\lambda| \leq \Lambda_{G>0}$:*

$$\boxed{\frac{1}{2} \leq H_n \leq \frac{3}{2}, \quad \forall n \geq 1.} \quad (153)$$

$|H_n - 1| = |\lambda|\sigma(4\omega n)^{\sigma-1}$. Since $\sigma - 1 < 0$, the function $(4\omega n)^{\sigma-1}$ is decreasing in n. Hence

$$(4\omega n)^{\sigma-1} \leq (4\omega\cdot 1)^{\sigma-1} = (4\omega)^{\sigma-1}, \qquad n \geq 1.$$

Therefore

$$|H_n - 1| \leq |\lambda|\sigma(4\omega)^{\sigma-1} \leq \Lambda_{G>0}\sigma(4\omega)^{\sigma-1} = \frac{1}{2\sigma(4\omega)^{\sigma-1}}\cdot\sigma(4\omega)^{\sigma-1} = \frac{1}{2}.$$

Thus $-1/2 \leq H_n - 1 \leq 1/2$, i.e. $1/2 \leq H_n \leq 3/2$.

Remark 3.23 (Explicit form for $\alpha = 1$ and $\sigma = 1/2$) *At $\alpha = 1$ and $\sigma = 1/2$: $H_n = 1 + \lambda/(4\sqrt{\omega n})$, $n \geq 1$.*

Remark 3.24 (Insufficiency of $\Lambda_{\text{gap}}^{\text{self}}$ for positivity) *Under the weaker hypothesis $|\lambda| \leq \Lambda_{\text{gap}}^{\text{self}}$ alone (without $\Lambda_{G>0}$), H_n may not be positive. The bound is*

$$|H_n - 1| \leq \Lambda_{\text{gap}}^{\text{self}}\sigma(4\omega)^{\sigma-1} = \frac{[\omega(2\mu+3)]^{1-\sigma}(4\omega)^{\sigma-1}}{4}.$$

For large μ, this may exceed $1/4$ (and hence H_n may be negative for small n).

Explicit numerical example. *Take $\mu = 100$, $\sigma = 1/2$, $\omega = 1$. Then:*

$$\Lambda_{\text{gap}}^{\text{self}} = \frac{[\omega(2\mu+3)]^{1-\sigma}}{4\sigma} = \frac{[203]^{1/2}}{4\cdot 1/2} = \frac{\sqrt{203}}{2} \approx 7.12;$$

$$\Lambda_{G>0} = \frac{1}{2\sigma(4\omega)^{\sigma-1}} = \frac{1}{2\cdot(1/2)\cdot 4^{-1/2}} = \frac{1}{4^{-1/2}} = 2;$$

$$\Lambda_{\text{gap}} = \min(7.12, 2) = 2.$$

At $\lambda = -5$ (which satisfies $|\lambda| = 5 \leq \Lambda_{\text{gap}}^{\text{self}} \approx 7.12$ but $> \Lambda_{G>0} = 2$):

$$H_1 = 1 + (-5)\cdot(1/2)\cdot 4^{-1/2} = 1 - 5/4 = -1/4 < 0.$$

This confirms that $\Lambda_{\text{gap}}^{\text{self}}$ alone does not guarantee positivity of H_n.

Remark 3.25 (Asymptotic behaviour of H_n) *As $n \to \infty$, $(4\omega n)^{\sigma-1} \to 0$ (since $\sigma - 1 < 0$), so $H_n \to 1$. This justifies using H_n as a "correction factor" close to 1 in the perturbative analysis.*

Lemma 3.26 (Effective energy gap) *Let $\mu > 1/2$, $\omega > 0$, $\sigma = 1/2$, and $|\lambda| \leq \Lambda_{\text{gap}}^{\text{self}}/2$. Then:*

$$\boxed{|e_{n+r} - e_n| \geq \frac{7\omega|r|}{2}, \quad \forall n \geq 0,\ r \neq 0.} \quad (154)$$

We estimate

$$|e_{n+r} - e_n| \geq |A_{n+r} - A_n| - |\lambda||A_{n+r}^{1/2} - A_n^{1/2}|.$$

First term. $|A_{n+r} - A_n| = 4\omega|r|$.
Second term. By the mean value theorem applied to $\phi(x) = x^{1/2}$ on the interval between A_n and A_{n+r}, there exists $\xi \in [\min(A_n, A_{n+r}), \max(A_n, A_{n+r})]$:

$$|A_{n+r}^{1/2} - A_n^{1/2}| = \frac{1}{2\sqrt{\xi}}|A_{n+r} - A_n| \leq \frac{1}{2\sqrt{A_0}}\cdot 4\omega|r| = \frac{2\omega|r|}{\sqrt{A_0}}.$$

With $|\lambda| \leq \sqrt{A_0}/4$: $|\lambda|\cdot 2\omega|r|/\sqrt{A_0} \leq (\sqrt{A_0}/4)\cdot 2\omega|r|/\sqrt{A_0} = \omega|r|/2$.
Combination. $|e_{n+r} - e_n| \geq 4\omega|r| - \omega|r|/2 = 7\omega|r|/2$.

Remark 3.27 (Relaxed bound $3\omega|r|$) *From Lemma 3.26, $|e_{n+r}-e_n| \geq 7\omega|r|/2 \geq 3\omega|r|$ for every $r \neq 0$ (since $7/2 = 3.5 > 3$). In Section 4.3, we use the weaker bound $2\omega|r| + \omega/2$ because $D_{n,r}(F)$ also includes F and the diagonal difference $\gamma(V^{\log}_{n+r,n+r} - V^{\log}_{nn})$. No contradiction between the bounds.*

3.3.6. Second-Order Expansion of the Eigenvalues

Theorem 3.28 (Second-order expansion) *Under hypotheses (H1)–(H3), for every $n \geq 1$, the n-th eigenvalue E_n of $\mathcal{H}(\lambda,\gamma)$ admits the expansion*

$$E_n = e_n + \gamma V^{\log}_{nn} + \gamma^2 E^{(2)}_n + O(\gamma^3), \tag{155}$$

where $e_n = A_n + \lambda A^{\sigma}_n$, and the second-order coefficient $E^{(2)}_n$ decomposes into an algebraic part and a logarithmic part:

$$\boxed{E^{(2)}_n = E^{(2),\mathrm{alg}}_n + E^{(2),\log}_n}, \tag{156}$$

with

$$E^{(2),\mathrm{alg}}_n = \boxed{-\frac{\lambda\pi^2\sigma(1-\sigma)}{24H_n^2}(4\omega n)^{\sigma-2} + O(n^{2\sigma-3})}, \tag{157}$$

$$E^{(2),\log}_n = \boxed{-\frac{\alpha(\log n + \gamma_E - \log 2)}{16\omega H^*_n(n+1)^2} + O(n^{-2})}, \tag{158}$$

where H_n and H^*_n are defined in Eq. (152) *(Definition 3.20).*

Convention on $\Delta_\pm$. Throughout this proof we set

$$\Delta^{(r)}_{\pm}|e_{n\pm r} - e_n| > 0, \qquad \text{i.e.} \quad \begin{aligned} \Delta^{(r)}_{+} &= e_{n+r} - e_n, \\ \Delta^{(r)}_{-} &= e_n - e_{n-r}. \end{aligned} \tag{159}$$

This convention is forced by the second-order perturbation formula

$$E^{(2)}_n = \sum_{m\neq n} \frac{|V^{\log}_{mn}|^2}{e_n - e_m}, \tag{160}$$

where the denominators $e_n - e_{n-r} = \Delta^{(r)}_{-} > 0$ and $e_n - e_{n+r} = -\Delta^{(r)}_{+} < 0$ have opposite signs.

Setup. Define

$$A_r|V^{\log}_{n,n-r}|^2, \qquad B_r|V^{\log}_{n,n+r}|^2. \tag{161}$$

Collecting the terms with $m = n - r$ and $m = n + r$ in Eq. (160):

$$E^{(2)}_n = \sum_{r\geq 1}\left[\frac{A_r}{\Delta^{(r)}_{-}} - \frac{B_r}{\Delta^{(r)}_{+}}\right]. \tag{162}$$

Algebraic decomposition. We split Eq. (162) into a symmetric and an antisymmetric part via the elementary identity

$$\frac{A_r}{\Delta_-} - \frac{B_r}{\Delta_+} = \frac{A_r + B_r}{2}\left(\frac{1}{\Delta_-} - \frac{1}{\Delta_+}\right) + \frac{A_r - B_r}{2}\left(\frac{1}{\Delta_-} + \frac{1}{\Delta_+}\right). \tag{163}$$

We therefore set

$$\begin{aligned} S^{\mathrm{alg}}_n \sum_{r\geq 1} \frac{A_r + B_r}{2}\left(\frac{1}{\Delta^{(r)}_{-}} - \frac{1}{\Delta^{(r)}_{+}}\right), \\ S^{\log}_n \sum_{r\geq 1} \frac{A_r - B_r}{2}\left(\frac{1}{\Delta^{(r)}_{-}} + \frac{1}{\Delta^{(r)}_{+}}\right), \end{aligned} \tag{164}$$

so that $E^{(2)}_n = S^{\mathrm{alg}}_n + S^{\log}_n$.

Part 1: Algebraic contribution S^{alg}_n.

Step 1 (Taylor expansion of $e_{n\pm r}$). With $h_r 4\omega r$ and $H^*_n 1 + \lambda\sigma A^{\sigma-1}_n$,

$$A^{\sigma}_{n+r} = A^{\sigma}_n + \sigma A^{\sigma-1}_n h_r + \frac{\sigma(\sigma-1)}{2}A^{\sigma-2}_n h^2_r + O(h^3_r A^{\sigma-3}_n), \tag{165}$$

$$A^{\sigma}_{n-r} = A^{\sigma}_n - \sigma A^{\sigma-1}_n h_r + \frac{\sigma(\sigma-1)}{2}A^{\sigma-2}_n h^2_r + O(h^3_r A^{\sigma-3}_n). \tag{166}$$

Taking the difference $e_{n\pm r} - e_n = \pm h_r + \lambda(A^{\sigma}_{n\pm r} - A^{\sigma}_n)$:

$$\Delta^{(r)}_{\pm} = 4\omega r H^*_n \pm \underbrace{\frac{\lambda\sigma(\sigma-1)}{2}A^{\sigma-2}_n(4\omega r)^2}_{=:C_r} + \underbrace{O(r^3 A^{\sigma-3}_n)}_{=:R_r}. \tag{167}$$

Set $D_r 4\omega r H^*_n$. Then $\Delta_\pm = D_r \pm C_r + R_r$ with $R_r = O(r^3 A^{\sigma-3}_n)$.

Step 2 (difference of inverses). By Taylor expansion of $1/(D_r \pm C_r + R_r)$:

$$\begin{aligned} \frac{1}{\Delta_-} - \frac{1}{\Delta_+} &= \frac{1}{D_r - C_r} - \frac{1}{D_r + C_r} + O(R_r/D^2_r) \\ &= \frac{2C_r}{D^2_r - C^2_r} + O(R_r/D^2_r) \\ &= \frac{2C_r}{D^2_r}\left(1 + \frac{C^2_r}{D^2_r} + O(A^{4\sigma-8}_n r^4)\right) + O(R_r/D^2_r). \end{aligned} \tag{168}$$

We compute the leading coefficient:

$$\frac{2C_r}{D^2_r} = \frac{\lambda\sigma(\sigma-1)A^{\sigma-2}_n(4\omega r)^2}{(4\omega r H^*_n)^2} = \frac{\lambda\sigma(\sigma-1)A^{\sigma-2}_n}{(H^*_n)^2}, \tag{169}$$

which is *independent of r*.

Step 3 (matrix element sum). By Lemma 3.12 and the uniform bound $|V_\pm|^2 \leq 1/(4r^2)$ of Theorem 3.10:

$$\begin{aligned} \frac{A_r + B_r}{2} &= \frac{1}{4r^2}(1 + O(r/n)) \quad (1 \leq r \leq n), \\ \sum_{r=1}^{\infty} \frac{A_r + B_r}{2} &= \frac{\pi^2}{24} + O(n^{-1}\log n). \end{aligned} \tag{170}$$

The logarithmic correction to the sum arises from $\sum_{r=1}^{n}(1/(4r^2)) \cdot \alpha r/n = (\alpha/(4n))\sum_{r=1}^{n} 1/r = O(n^{-1}\log n)$.

Step 4 (leading term). Substituting Eq. (169) and Eq. (170) into S^{alg}_n:

$$S^{\mathrm{alg,leading}}_n = \frac{\lambda\sigma(\sigma-1)A^{\sigma-2}_n}{(H^*_n)^2} \cdot \frac{\pi^2}{24} + O(n^{\sigma-3}\log n), \tag{171}$$

where the error term absorbs the product of the coefficient $|\lambda\sigma(\sigma-1)|A_n^{\sigma-2}/(H_n^*)^2 = O(n^{\sigma-2})$ with the $O(n^{-1}\log n)$ error in the matrix-element sum Eq. (170), giving $O(n^{\sigma-2}\cdot n^{-1}\log n) = O(n^{\sigma-3}\log n)$.

Step 5 (subleading corrections). The first subleading correction in Eq. (168) is $\frac{2C_r}{D_r^2}\cdot\frac{C_r^2}{D_r^2} = \frac{2C_r^3}{D_r^4}$. We estimate its contribution:

$$\frac{2C_r^3}{D_r^4} = \frac{2C_r}{D_r^2}\cdot\frac{C_r^2}{D_r^2} = O(A_n^{\sigma-2})\cdot O(r^2A_n^{2\sigma-4}) = O(r^2A_n^{3\sigma-6}). \tag{172}$$

Multiplying by $(A_r+B_r)/2 = O(1/r^2)$ and summing over r:

$$\sum_{r\geq 1}\frac{A_r+B_r}{2}\cdot\frac{2C_r^3}{D_r^4} = O(A_n^{3\sigma-6})\sum_{r=1}^{n}1 = O(n^{3\sigma-5}). \tag{173}$$

Since $3\sigma-5<2\sigma-3$ for all $\sigma\in(0,1)$, this subleading correction is absorbed into the stated error $O(n^{2\sigma-3})$.

Step 6 (final form of S_n^{alg}). From Eq. (171) and Eq. (173), and using $H_n^* = H_n + O(1/n)$ and $A_n^{\sigma-2} = (4\omega n+\omega(2\mu+3))^{\sigma-2} = (4\omega n)^{\sigma-2}(1+O(1/n))$:

$$\boxed{S_n^{\mathrm{alg}} = -\frac{\lambda\pi^2\sigma(1-\sigma)}{24H_n^2}(4\omega n)^{\sigma-2} + O(n^{2\sigma-3}).} \tag{174}$$

Part 2: Logarithmic contribution $S_n^{\log}$.

Step 1 (sum of inverses). From Eq. (168) and the parallel computation for the sum:

$$\frac{1}{\Delta_-}+\frac{1}{\Delta_+} = \frac{1}{D_r-C_r}+\frac{1}{D_r+C_r}+O(R_r/D_r^2) = \frac{2D_r}{D_r^2-C_r^2}+O(R_r/D_r^2). \tag{175}$$

We compute:

$$\frac{2D_r}{D_r^2-C_r^2} = \frac{2}{D_r}\left(1+\frac{C_r^2}{D_r^2}+\dots\right) = \frac{1}{2\omega rH_n^*}\left(1+O(r^2A_n^{2\sigma-4})\right). \tag{176}$$

Step 2 (asymmetry of matrix elements). By Lemma 3.15, for $n\geq n_\Delta^{(4)}$ and $1\leq r\leq\lfloor n/2\rfloor$:

$$\delta_r^{\mathrm{mat}} = A_r - B_r = -\frac{\alpha}{4n^2}(1+O(\alpha r/n)). \tag{177}$$

For $r>\lfloor n/2\rfloor$, we use the cruder bound from Theorem 3.10: $|\delta_r^{\mathrm{mat}}|\leq 1/(2r^2)$, which contributes $O(n^{-2})$ to $S_n^{\log}$.

Step 3 (main contribution). Combining Steps 1 and 2:

$$S_n^{\log} = \sum_{r=1}^{\lfloor n/2\rfloor}\frac{\delta_r^{\mathrm{mat}}}{2}\cdot\frac{1}{2\omega rH_n^*}\left(1+O(r^2A_n^{2\sigma-4})\right)+O(n^{-2})$$
$$= -\frac{\alpha}{16\omega H_n^*n^2}\sum_{r=1}^{\lfloor n/2\rfloor}\frac{1}{r}(1+O(\alpha r/n)+ \tag{178}$$
$$O(r^2A_n^{2\sigma-4})+O(n^{-2}). \tag{179}$$

Step 4 (error analysis). The two error terms contribute:

- $\sum_r\frac{\alpha r/n}{r} = \frac{\alpha}{n}\cdot\frac{n}{2} = O(\alpha)$. After the prefactor $\alpha/(16\omega H_n^*n^2)$: $O(n^{-2})$.
- $\sum_r\frac{r^2A_n^{2\sigma-4}}{r} = A_n^{2\sigma-4}\sum_{r=1}^{n/2}r = O(n^{2\sigma-2})$. After the prefactor: $O(n^{2\sigma-4})$. For $\sigma\in(0,1)$: $2\sigma-4<0$, dominated by $O(n^{-2})$ at leading order.

Hence the error terms are absorbed into $O(n^{-2})$.

Step 5 (harmonic sum). The key identity:

$$\sum_{r=1}^{\lfloor n/2\rfloor}\frac{1}{r} = H_{\lfloor n/2\rfloor} = \log(n/2)+\gamma_E+O(1/n) = \log n+\gamma_E-\log 2+O(1/n). \tag{180}$$

The factor $-\log 2$ arises from the upper limit $\lfloor n/2\rfloor$ of the sum; it would be absent if the sum extended to n.

Step 6 (final form of $S_n^{\log}$). Combining Steps 3–5:

$$\boxed{S_n^{\log} = -\frac{\alpha(\log n+\gamma_E-\log 2)}{16\omega H_n^*(n+1)^2}+O(n^{-2}).} \tag{181}$$

The shift from n^2 to $(n+1)^2$ in the denominator is a harmless reparametrization that absorbs $O(n^{-3})$ corrections.

Part 3: Combination and remainder.

Step 1 (combination). Adding Eq. (174) and Eq. (181):

$$E_n^{(2)} = S_n^{\mathrm{alg}}+S_n^{\log} = -\frac{\lambda\pi^2\sigma(1-\sigma)}{24H_n^2}(4\omega n)^{\sigma-2} - \frac{\alpha(\log n+\gamma_E-\log 2)}{16\omega H_n^*(n+1)^2}+O(n^{2\sigma-3})+O(n^{-2}). \tag{182}$$

This proves Eq. (157)–Eq. (158).

Step 2 (third-order remainder). The remainder $O(\gamma^3)$ involves the third-order coefficient

$$\Sigma_n^{(3)}\sum_{m,k\neq n}\frac{V_{nm}^{\log}V_{mk}^{\log}V_{kn}^{\log}}{(e_n-e_m)(e_n-e_k)} - V_{nn}^{\log}\sum_{m\neq n}\frac{|V_{nm}^{\log}|^2}{(e_n-e_m)^2}. \tag{183}$$

We bound the two terms separately. Using $|V_{nm}^{\log}|\leq 1/(2|m-n|)$ (Theorem 3.10) and $|e_n-e_m|\geq 2\omega|m-n|$ (Lemma 3.26), we write $p=n-m$ and $r=n-k$.

(a) Triple convolution term. The first term in Eq. (183) satisfies

$$|\Sigma_n^{(3),\mathrm{triple}}|\leq\frac{1}{32\omega^2}\sum_{m,k\neq n}\frac{1}{|n-m|^2|m-k||n-k|^2}.$$

Writing $p=n-m$, $r=n-k$, the inner sum evaluates to

$$\sum_{r\neq 0,r\neq p}\frac{1}{|r|^2|r-p|} = O(1/|p|),$$

by partial-fraction decomposition of the composed Hilbert kernel (equivalently, by iterated application of the discrete Hilbert transform bound of [36, Ch. IX, Theorem 316]). The outer sum then gives

$$|\Sigma_n^{(3),\mathrm{triple}}|\leq\frac{C_1}{\omega^2}\sum_{p\neq 0}\frac{1}{|p|^3} = O(1/\omega^2).$$

(b) Diagonal term. The second term in Eq. (183) satisfies

$$|\Sigma_n^{(3),\mathrm{diag}}| \le |V_{nn}^{\log}| \cdot \frac{1}{16\omega^2} \sum_{m\neq n} \frac{1}{|n-m|^4}.$$

By Lemma 3.7, $|V_{nn}^{\log}| = \frac{1}{2}|\Psi(n+\alpha+1) - \log\omega|$. Using the asymptotic $\Psi(z) = \log z + O(1/z)$ as $z \to \infty$ and $|\log\omega| \le \log n$ for $\omega \ge 1/\sqrt{n}$ (which holds for large n), we have $|V_{nn}^{\log}| \le C\log n$ for all sufficiently large n. Also $\sum_{m\neq n}|n-m|^{-4} = 2\zeta(4) = \pi^4/45 < \infty$. Hence $|\Sigma_n^{(3),\mathrm{diag}}| = O(\log n/\omega^2)$.

(c) Combination. Combining (a) and (b), $|\Sigma_n^{(3)}| \le C\log n/\omega^2$ for a universal constant C independent of n and ω (for n sufficiently large). The logarithmic factor originates solely from the diagonal term $V_{nn}^{\log}$, not from the triple convolution. Hence $\gamma^3 E_n^{(3)} = O(\gamma^3 \log n/\omega^2)$.

Remark on uniformity in n. The remainder $O(\gamma^3\log n/\omega^2)$ is uniform in γ for fixed n but grows logarithmically in n. A sharper, n-uniform bound on the second-order and higher-order corrections is provided by the fixed-point formulation in Section 4.3.

Corollary 3.29 (Explicit form for $\sigma = 1/2$) *At $\sigma = 1/2$:*

$$E_n^{(2),\mathrm{alg}} = \boxed{-\frac{\lambda\pi^2}{768\,\omega^{3/2}H_n^2}\,n^{-3/2} + O(n^{-5/2}\log n)}, \tag{184}$$

$$E_n^{(2),\log} = \boxed{-\frac{\alpha(\log n + \gamma_E - \log 2)}{16\omega H_n^*(n+1)^2} + O(n^{-2})}. \tag{185}$$

At $\sigma = 1/2$: $\sigma(1-\sigma) = 1/4$ and $(4\omega n)^{-3/2} = \frac{1}{8}\omega^{-3/2}n^{-3/2}$. Substituting into Eq. (157):

$$-\frac{\lambda\pi^2}{24H_n^2}\cdot\frac{1}{4}\cdot\frac{1}{8}\omega^{-3/2}n^{-3/2} = -\frac{\lambda\pi^2}{768\,\omega^{3/2}H_n^2}n^{-3/2}.$$

The error term is refined from the general $O(n^{2\sigma-3}) = O(n^{-2})$ to $O(n^{-5/2}\log n)$ by the refined analysis in Part 1, Step 3 (logarithmic correction to the sum of matrix elements). The form of Eq. (185) is identical to Eq. (158).

Remark 3.30 (Refined error bound and conservatism of Eq. (157)]

The error bound $O(n^{2\sigma-3})$ in Eq. (157) is conservative and uniform in $\sigma \in (0,1)$. A closer examination of the sources of error in Part 1 of the proof of Theorem 3.28 reveals the following hierarchy, ordered from largest to smallest:

- *Dominant error:* the $O(r/n)$ correction in the expansion of $(A_r+B_r)/2$ in Eq. (170), together with the Taylor truncation error from $R_r = O(r^3 A_n^{\sigma-3})$, contributing $O(n^{\sigma-3}\log n)$ to S_n^{alg};
- *First subleading error:* the correction from the Taylor remainder R_r, given by $C_r R_r/D_r^3 = O(n^{2\sigma-4})$;
- *Second subleading error:* the C_r^3/D_r^4 correction in Eq. (168), contributing $O(n^{3\sigma-5})$, as computed in Eq. (173).

For every $\sigma \in (0,1)$, all three errors are strictly smaller than the stated bound $O(n^{2\sigma-3})$:

$$\sigma - 3 < 2\sigma - 3 \quad (\text{since } \sigma > 0), \qquad 2\sigma-4 < 2\sigma-3,$$
$$3\sigma - 5 < 2\sigma - 3 \quad (\text{since } \sigma < 2).$$

We retain the conservative bound $O(n^{2\sigma-3})$ in the statement of Theorem 3.28 because it is uniform in σ without logarithmic factors and because all applications in this paper use only the sign and leading order of $E_n^{(2),\mathrm{alg}}$, not the precise subleading rate. At $\sigma = 1/2$, the sharp bound $O(n^{-5/2}\log n)$ is stated explicitly in Corollary 3.29.

Remark 3.31 (Hierarchy of the two terms) *For $\sigma > 1/2$, the algebraic term dominates (exponent $\sigma - 2 < -1$). For $\sigma = 1/2$, both terms are of comparable order ($n^{-3/2}$ vs $n^{-2}\log n$), with the algebraic term dominating for large n. The algebraic term vanishes if and only if $\lambda = 0$; the logarithmic term is present for all λ (including $\lambda = 0$) and is proportional to $\alpha = \mu + 1/2$.*

4. CONVOLUTION OPERATOR AND FIXED-POINT FORMULATION

In this section we develop the analytical tools needed to formulate the eigenvalue problem for $\mathcal{H}(\lambda,\gamma)$ as a fixed-point equation on a weighted Banach space, and we prove existence and uniqueness of the solution. All constants are explicit, and all proofs are self-contained. The section is organized as follows: §4.1 establishes the diagonal-difference estimate; §4.2 introduces the convolution operator T and proves its boundedness via the Schur test and Stein interpolation, together with the Extended Convolution Lemma; §4.3 develops the fixed-point formulation for $n \ge 1$; and §4.4 treats the case $n = 0$ separately due to the index constraint $s \ge 1$.

4.1. Diagonal Difference Estimate

Lemma 4.1 (Diagonal difference bound) *There exists an explicit constant*

$$\boxed{C_V\,\frac{1}{2}\left(1 + \frac{1}{2\log 2}\right) \approx 0.8607,} \tag{186}$$

such that for every $n \ge 0$ and $r \neq 0$ with $n + r \ge 0$:

$$|V_{n+r,n+r}^{\log} - V_{nn}^{\log}| \le C_V \log(2+|r|). \tag{187}$$

Step 1 (exact expression). By Lemma 3.7, for every $k \ge 0$:

$$V_{kk}^{\log} = \frac{1}{2}[\Psi(k+\alpha+1) - \log\omega], \tag{188}$$

where $\Psi = \Gamma'/\Gamma$ is the digamma function. Taking the difference between $k = n + r$ and $k = n$:

$$V^{\log}_{n+r,n+r} - V^{\log}_{nn} = \frac{1}{2}[\Psi(a+r) - \Psi(a)], \qquad an + \alpha + 1. \tag{189}$$

It suffices to prove the bound for $r > 0$ (the case $r < 0$ follows by antisymmetry of $\Psi(a+r) - \Psi(a)$).

Step 2 (lower bound for a). Since $n \geq 0$ and $\alpha = \mu + 1/2 > 1$: $a \geq \alpha + 1 > 2$.

Step 3 (digamma asymptotic). From NIST DLMF [37, §5.11(i), eq. 5.11.2]:

$$\Psi(z) = \log z - \frac{1}{2z} - \sum_{m=1}^{\infty} \frac{B_{2m}}{2mz^{2m}}, \qquad z > 0, \tag{190}$$

with Bernoulli numbers $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$. Subtracting the expansions for $z = a + r$ and $z = a$:

$$\Psi(a+r) - \Psi(a) = \log \frac{a+r}{a} - \frac{1}{2(a+r)} + \frac{1}{2a} + S(a,r), \tag{191}$$

where

$$S(a,r) \sum_{m=1}^{\infty} \frac{B_{2m}}{2m} \left(\frac{1}{a^{2m}} - \frac{1}{(a+r)^{2m}} \right). \tag{192}$$

Step 4 (bound on $S(a,r)$). Since $a + r > a > 0$, we have $a^{-2m} > (a+r)^{-2m}$ for every m, so each term is non-negative. Hence

$$|S(a,r)| \leq \sum_{m=1}^{\infty} \frac{|B_{2m}|}{2ma^{2m}}. \tag{193}$$

Using $a > 2$ (so $a^{-2} < 1/4$) and the numerical values above:

$$\begin{aligned} |S(a,r)| &\leq \frac{1}{12a^2} + \frac{1}{120a^4} + \frac{1}{252a^6} + \frac{1}{240a^8} + \cdots \\ &\leq \frac{1}{48} + \frac{1}{1920} + \frac{1}{16128} + \frac{1}{61440} + \cdots \\ &= 0.0208333 + 0.0005208 + 0.0000620 + \\ &\quad 0.0000163 + \cdots < 0.0215 \leq \frac{1}{4}. \end{aligned} \tag{194}$$

Step 5 (individual bounds).

- Logarithmic term: $\log((a+r)/a) = \log(1 + r/a) \leq \log(1+r) \leq \log(2+r)$;
- $\frac{1}{2a} - \frac{1}{2(a+r)} = \frac{r}{2a(a+r)} \leq \frac{1}{2a} \leq \frac{1}{4}$ (since $r/(a+r) < 1$ and $a > 2$);
- $S(a,r)$: $|S(a,r)| \leq 1/4$.

Step 6 (combination). For $r > 0$:

$$|\Psi(a+r) - \Psi(a)| \leq \log(2+r) + \frac{1}{4} + \frac{1}{4} = \log(2+r) + \frac{1}{2}. \tag{195}$$

Therefore

$$|V^{\log}_{n+r,n+r} - V^{\log}_{nn}| \leq \frac{1}{2}\log(2+r) + \frac{1}{4}. \tag{196}$$

Step 7 (conversion to C_V). Since $\log(2 + |r|) \geq \log 2$ for $|r| \geq 0$:

$$|V^{\log}_{n+r,n+r} - V^{\log}_{nn}| \leq \frac{1}{2}\log(2+r) + \frac{\log(2+r)}{4\log 2} = C_V \log(2+|r|), \tag{197}$$

where $C_V = \frac{1}{2}(1 + \frac{1}{2\log 2}) \approx 0.8607$.

4.2. Convolution Operator and Weighted Sequence Spaces

4.2.1. Weighted Sequence Spaces

Definition 4.2 (Space Y_α) *For $\alpha \in \mathbb{R}$:*

$$Y_\alpha \left\{ u = (u_r)_{r\neq 0} : \|u\|^2_{Y_\alpha} \sum_{r\neq 0} |r|^\alpha |u_r|^2 < \infty \right\}, \tag{198}$$

with inner product $\langle u, v\rangle_{Y_\alpha} \sum_{r\neq 0} |r|^\alpha u_r \bar{v}_r$.

Definition 4.3 (Space Z'_η) *For $\eta > 0$:*

$$Z'_\eta \left\{ u = (u_r)_{r\neq 0} : \|u\|_{Z'_\eta} \sup_{r\neq 0} \frac{|r|^\eta}{(\log(2+|r|))^\eta} |u_r| < \infty \right\}. \tag{199}$$

This is a Banach space (closed subspace of ℓ^∞ after weight transformation).

Lemma 4.4 (Embedding $Z'_2 \subset Y_2$) *There exists a constant $C_{\text{emb}} \approx 583$ such that*

$$\|u\|_{Y_2} \leq C_{\text{emb}} \|u\|_{Z'_2}, \qquad \forall u \in Z'_2. \tag{200}$$

For $u \in Z'_2$ and every $r \neq 0$:

$$|u_r| \leq \|u\|_{Z'_2} \frac{(\log(2+|r|))^2}{|r|^2}.$$

Squaring and summing:

$$\|u\|^2_{Y_2} = \sum_{r\neq 0} |r|^2 |u_r|^2 \leq \|u\|^2_{Z'_2} \sum_{r\neq 0} \frac{(\log(2+|r|))^4}{|r|^2}. \tag{201}$$

The summand is even, so

$$\sum_{r\neq 0} \frac{(\log(2+|r|))^4}{|r|^2} = 2\sum_{r=1}^{\infty} \frac{(\log(2+r))^4}{r^2}. \tag{202}$$

Using the elementary inequality $(\log s)^4 \leq (4/(e\epsilon))^4 s^\epsilon$ for $s \geq 1$ and $\epsilon > 0$, with $\epsilon = 1/4$:

$$(\log(2+t))^4 \leq 16^4 \cdot (2+t)^{1/4} \leq 65536 \cdot 3^{1/4}. \\ t^{1/4}, \qquad t \geq 1.$$

Hence

$$\sum_{r\neq 0} \frac{(\log(2+|r|))^4}{|r|^2} \leq 2 \cdot 65536 \cdot 3^{1/4} \cdot \zeta(7/4) \approx 2 \cdot 65536 \cdot \\ 1.3161 \cdot 1.9680 \approx 3.395 \times 10^5.$$

Therefore

$$\|u\|_{Y_2} \leq \sqrt{3.395 \times 10^5} \|u\|_{Z'_2} \approx 582.7 \|u\|_{Z'_2} \leq 583 \|u\|_{Z'_2}.$$

✓

4.2.2. Working Constants and Convolution Operator

Definition 4.5 (Working constants) $c_\omega 2\omega$, $R\omega/2$.

Definition 4.6 (Convolution operator) *On sequences* $u = (u_r)_{r\neq 0}$*:*

$$(Tu)_r \sum_{\substack{s\neq 0\\ s\neq r}} \frac{1}{|r||r-s|} u_s, \qquad r \neq 0, \tag{203}$$

with $T_{rs} = 1/(|r||r-s|)$ *for* $r \neq s$ *and* $T_{rr} = 0$.

4.2.3. Schur Test and Stein Interpolation

Theorem 4.7 (Schur test) *Let* $K = (K_{rs})$ *be a non-negative kernel on a countable set* I *and* $p_r > 0$ *weights. If* $C_1, C_2 < \infty$ *with*

$$\sum_{s\in I} K_{rs} p_s \le C_1 p_r, \qquad \sum_{r\in I} K_{rs} p_r \le C_2 p_s, \qquad \forall r,s \in I, \tag{204}$$

then $\|K\|_{\ell^2\to\ell^2} \le \sqrt{C_1 C_2}$.

For $u \in \ell^2(I)$ and every $r \in I$:

$$|(Ku)_r|^2 = \left|\sum_s K_{rs}^{1/2} p_s^{1/2} \cdot K_{rs}^{1/2} \frac{u_s}{p_s^{1/2}}\right|^2 \le \left(\sum_s K_{rs} p_s\right) \left(\sum_s K_{rs} \frac{|u_s|^2}{p_s}\right) \le C_1 p_r \sum_s K_{rs} \frac{|u_s|^2}{p_s}.$$

Summing over r and using Fubini (all terms non-negative):

$$\|Ku\|_{\ell^2}^2 \le C_1 \sum_s \frac{|u_s|^2}{p_s} \sum_r K_{rs} p_r \le C_1 C_2 \sum_s |u_s|^2 = C_1 C_2 \|u\|_{\ell^2}^2.$$

Taking square roots gives the result. ✓

Theorem 4.8 (Stein interpolation) *Let* $S = \{z \in \mathbb{C} : 0 \le \Re z \le 1\}$ *and* $\{T_z\}_{z\in S}$ *analytic with* $\|T_{i\tau}\|_{\ell^2\to\ell^2} \le M_0$ *and* $\|T_{1+i\tau}\|_{\ell^2\to\ell^2} \le M_1$ *for all* $\tau \in \mathbb{R}$*. Then* $\|T_\theta\|_{\ell^2\to\ell^2} \le M_0^{1-\theta} M_1^\theta$.

See [30, Chapter V, Theorem 1.4] or [31, Section 1.18].

4.2.4. Closed Forms for $F(r)$ and $G(r)$

Lemma 4.9 (Closed forms for $F(r)$ and $G(r)$) *For every* $r \ge 1$*:*

$$F(r) \sum_{\substack{s\neq 0\\ s\neq r}} \frac{1}{s^2|r-s|} = \frac{2H_r}{r^2} + \frac{2H_r^{(2)}}{r} - \frac{3}{r^3}, \tag{205}$$

$$\boxed{G(r) \sum_{\substack{s\neq 0\\ s\neq r}} \frac{1}{|s||r-s|} = \frac{4H_r}{r} - \frac{2}{r^2},} \tag{206}$$

where $H_r = \sum_{k=1}^r 1/k$ *and* $H_r^{(2)} = \sum_{k=1}^r 1/k^2$.

Part A: $G(r)$**.** Split the sum into three regions.

Region I ($1 \le s \le r-1$)*.* Using $\frac{1}{s(r-s)} = \frac{1}{r}(\frac{1}{s} + \frac{1}{r-s})$:

$$G_{\rm I} = \frac{1}{r}\sum_{s=1}^{r-1}\left(\frac{1}{s} + \frac{1}{r-s}\right) = \frac{2H_{r-1}}{r}.$$

Region II ($s \ge r+1$)*.* Using $\frac{1}{s(s-r)} = \frac{1}{r}(\frac{1}{s-r} - \frac{1}{s})$:

$$G_{\rm II} = \frac{1}{r}\sum_{s=r+1}^{\infty}\left(\frac{1}{s-r} - \frac{1}{s}\right) = \frac{1}{r}\lim_{N\to\infty}\sum_{s=r+1}^{N}\left(\frac{1}{s-r} - \frac{1}{s}\right).$$

The finite sum telescopes:

$$\sum_{s=r+1}^{N}\left(\frac{1}{s-r} - \frac{1}{s}\right) = \sum_{j=1}^{N-r}\frac{1}{j} - \sum_{s=r+1}^{N}\frac{1}{s} = H_{N-r} - (H_N - H_r).$$

As $N \to \infty$: $H_{N-r} - H_N = -\sum_{j=N-r+1}^{N} 1/j \to 0$ (each term is $O(1/N)$, and there are r terms). Hence $G_{\rm II} = H_r/r$.

Region III ($s = -k$, $k \ge 1$)*.* Using $\frac{1}{k(r+k)} = \frac{1}{r}(\frac{1}{k} - \frac{1}{r+k})$:

$$G_{\rm III} = \frac{1}{r}\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{r+k}\right) = \frac{H_r}{r}.$$

Combination.

$$G(r) = \frac{2H_{r-1}}{r} + \frac{H_r}{r} + \frac{H_r}{r} = \frac{2H_{r-1} + 2H_r}{r} = \frac{4H_r - 2/r}{r} = \frac{4H_r}{r} - \frac{2}{r^2}. \quad ✓$$

Part B: $F(r)$**.**

Region I ($1 \le s \le r-1$)*.* Write $\frac{1}{s^2(r-s)} = \frac{1}{r}\left(\frac{1}{s^2} + \frac{1}{s(r-s)}\right)$:

$$F_{\rm I} = \frac{H_{r-1}^{(2)}}{r} + \frac{2H_{r-1}}{r^2}.$$

Region II ($s \ge r+1$)*.* Write $s = r+j$ with $j \ge 1$. Using the partial fraction $\frac{1}{(r+j)^2 j} = \frac{1}{r^2}\left(\frac{1}{j} - \frac{1}{r+j}\right) - \frac{1}{r(r+j)^2}$:

$$F_{\rm II} = \frac{H_r}{r^2} - \frac{1}{r}\left(\frac{\pi^2}{6} - H_r^{(2)}\right).$$

Region III ($s = -k$, $k \ge 1$)*.* Write $\frac{1}{k^2(r+k)} = \frac{1}{r}\left(\frac{1}{k^2} - \frac{1}{rk} + \frac{1}{r(r+k)}\right)$:

$$F_{\rm III} = \frac{\pi^2}{6r} - \frac{H_r}{r^2}.$$

Combination. Adding:

$$\begin{aligned} F(r) &= \frac{H_{r-1}^{(2)} + H_r^{(2)}}{r} + \frac{2H_{r-1}}{r^2} + \left(-\frac{\pi^2}{6r} + \frac{\pi^2}{6r}\right) \\ &= \frac{2H_r^{(2)} - 1/r^2}{r} + \frac{2H_r - 2/r}{r^2} \\ &= \frac{2H_r}{r^2} + \frac{2H_r^{(2)}}{r} - \frac{3}{r^3}. \quad ✓ \end{aligned}$$

Remark 4.10 (Numerical verification) *At* $r = 2$*:* $H_2 = 3/2$, $H_2^{(2)} = 5/4$. $G(2) = 4\cdot(3/2)/2 - 2/4 = 3 - 0.5 = 2.5$. $F(2) = 2(3/2)/4 + 2(5/4)/2 - 3/8 = 3/4 + 5/4 - 3/8 = 13/8 = 1.625$.

4.2.5. Sharp Tail Estimate

Lemma 4.11 (Sharp tail estimate) *For every $\eta > 1$ and $M \geq 1$:*

$$\sum_{s \geq M} \frac{(\log(2+s))^\eta}{s^{\eta+1}} \leq C_\eta^{\text{tail}} \frac{(\log(2+M))^\eta}{M^\eta}. \quad (207)$$

At $\eta = 2$: $C_2^{\text{tail}} \leq 2$.

We prove the case $\eta = 2$. Let $f(t) = (\log(2+t))^2/t^3$.

Step 1 (integration by parts). With $u = (\log(2+t))^2$ and $dv = t^{-3}dt$:

$$\int_M^\infty f = \frac{(\log(2+M))^2}{2M^2} + \int_M^\infty \frac{\log(2+t)}{t^2(2+t)}\,dt. \quad (208)$$

Step 2 (bounding the remaining integral). Since $1/(2+t) \leq 1/t$ for $t \geq 0$:

$$\int_M^\infty \frac{\log(2+t)}{t^2(2+t)}dt \leq \int_M^\infty \frac{\log(2+t)}{t^3}dt$$
$$\leq \frac{\log(2+M)}{2M^2} + \frac{1}{4M^2}.$$

Step 3 (combination).

$$\int_M^\infty f \leq \frac{(\log(2+M))^2}{2M^2} + \frac{\log(2+M)}{2M^2} + \frac{1}{4M^2}. \quad (209)$$

Step 4 (integral comparison). Since f is decreasing for $t \geq 1$:

$$\sum_{s \geq M} f(s) \leq f(M) + \int_M^\infty f.$$

Step 5 (case $M \geq 2$). Substituting:

$$\sum_{s \geq M} f(s) \leq \frac{(\log(2+M))^2}{M^3} + \frac{(\log(2+M))^2}{2M^2} + \frac{\log(2+M)}{2M^2} + \frac{1}{4M^2}.$$

Dividing by $(\log(2+M))^2/M^2$:

$$R(M) \leq \frac{1}{M} + \frac{1}{2} + \frac{1}{2\log(2+M)} + \frac{1}{4(\log(2+M))^2} \leq 1.4908 < 2.$$

Step 6 (case $M = 1$). Partial sum: $1.20695 + 0.24023 + 0.09594 = 1.54312$. Tail estimate from $s = 4$: $0.05016 + 0.17195 = 0.22211$. Total: $1.76523 < 2(\log 3)^2 = 2.41390$. ✓

Lemma 4.12 (Improved J_r bound) *For every $r \geq 1$:*

$$J_r \sum_{r/2<|s|<2r, s\neq r} \frac{1}{|r-s|} \leq 1.05r. \quad (210)$$

Split $J_r = J_r^+ + J_r^-$.

Positive part. Split into two groups:

- Group A: $r/2 < s < r$. Set $m = r - s$, so $1 \leq m \leq r - \lfloor r/2 \rfloor - 1$.
- Group B: $r < s < 2r$. Set $m = s - r$, so $1 \leq m \leq r - 1$.

Hence $J_r^+ = H_{r-\lfloor r/2\rfloor-1} + H_{r-1}$.

Negative part. With $s = -k$, $r/2 < k < 2r$:

$$J_r^- = \sum_{r/2<k<2r} \frac{1}{r+k} \leq \log 2.$$

Direct verification for $r \in \{1, \ldots, 10\}$:

r	J_r^+	J_r^-	J_r (vs. $1.05r$)
1	0.5	0	$0.5 \leq 1.05$
2	1.25	0.45	$1.45 \leq 2.10$
3	2.5	0.6345	$3.1345 \leq 3.15$
4	2.8333	0.5699	$3.4032 \leq 4.20$
5	3.5833	0.6587	$4.2420 \leq 5.25$
6	3.7833	0.6105	$4.3939 \leq 6.30$
7	4.2833	0.6687	$4.9521 \leq 7.35$
8	4.4262	0.6311	$5.0573 \leq 8.40$
9	4.8012	0.6743	$5.4755 \leq 9.45$
10	4.9123	0.6434	$5.5557 \leq 10.50$

General case $r \geq 11$. Using $H_m \leq \log m + \gamma_E + 1/(2m)$:

$$J_r \leq 2\log r + 2\gamma_E + \frac{1}{2(r - \lfloor r/2 \rfloor - 1)} + \frac{1}{2(r-1)} + \log 2 \leq 2\log r + 1.1544 + 0.1 + 0.05 + 0.6932$$
$$= 2\log r + 1.9976.$$

At $r = 11$: $6.7934 \leq 11.55$. The function $g(r) = 1.05r - 2\log r - 1.9976$ is increasing for $r \geq 2$, and $g(11) > 0$, so $g(r) > 0$ for all $r \geq 11$. ✓

4.2.6. The Quantities S_2 and $\widetilde{S}_2$

Lemma 4.13 (The quantity $\widetilde{S}_2$)

$$\widetilde{S}_2 \sum_{s\geq 1} \frac{(\log(2+s))^2}{s^3} < 1.80. \quad (211)$$

Partial sum to $s = 5$: 1.62357. Tail from $s = 6$: 0.14786. Total: $1.77143 < 1.80$. ✓

Lemma 4.14 (The quantity S_2)

$$S_2 2 \sum_{s\geq 1} \frac{(\log(2+s))^2}{s^2} < 8.92. \quad (212)$$

Partial sum to $s = 10$: 2.76132. Tail from $s = 11$: 1.52700. Total: 4.28832, hence $S_2 = 8.57664 < 8.92$. ✓

Remark 4.15 (Distinction between S_2 and $\widetilde{S}_2$) *S_2 and $\widetilde{S}_2$ differ in the power of s in the denominator.*

4.2.7. Extended Convolution Lemma

Lemma 4.16 (Extended Convolution Lemma) *For every $\eta > 1$ and $r \neq 0$:*

$$\sum_{\substack{s\neq 0\\ s\neq r}} \frac{(\log(2+|s|))^\eta}{|s|^\eta |r-s|} \leq C_\eta^B \frac{(\log(2+|r|))^\eta}{|r|}. \quad (213)$$

At $\eta = 2$: $C_2^B \leq 22.4$.

Fix $r > 0$. Split into three regions.
Region 1 ($|s| \le r/2$). Then $|r-s| \ge r/2$:

$$\Sigma_2^{(1)}(r) \le \frac{2S_2}{r} \le \frac{2\cdot 8.92}{(\log 3)^2}\cdot\frac{(\log(2+r))^2}{r} = 14.78\cdot\frac{(\log(2+r))^2}{r}.$$

Hence $C_2^{(1)} \le 14.78 \le 15$.
Region 2 ($r/2 < |s| < 2r$, $s \ne r$). $g(x) = (\log(2+x))^2/x^2$ is decreasing, so for $|s| > r/2$:

$$\Sigma_2^{(2)}(r) \le \frac{4(\log(2+r))^2}{r^2}\cdot J_r \le \frac{4\cdot 1.05(\log(2+r))^2}{r} = 4.2\cdot\frac{(\log(2+r))^2}{r}.$$

Hence $C_2^{(2)} \le 4.2$.
Region 3 ($|s| \ge 2r$). Then $|r-s| \ge |s|/2$:

$$\Sigma_2^{(3)}(r) \le 4\sum_{s\ge 2r}\frac{(\log(2+s))^2}{s^3} \le 4\cdot 2\cdot\frac{(\log(2+2r))^2}{(2r)^2} = \frac{2(\log(2+2r))^2}{r^2}.$$

The ratio $(\log(2+2r))^2/(r(\log(2+r))^2)$ is decreasing for $r \ge 1$; at $r=1$: $(\log 4)^2/(\log 3)^2 = 1.5923$. Hence

$$\Sigma_2^{(3)}(r) \le \frac{2\cdot 1.5923(\log(2+r))^2}{r} = 3.1846\cdot\frac{(\log(2+r))^2}{r}.$$

Hence $C_2^{(3)} \le 3.1846 \le 3.19$.
Combination. $C_2^B \le 14.78 + 4.2 + 3.19 = 22.17 \le 22.4$. ✓

Remark 4.17 (Conservative bound) *We adopt $C_2^B \le 22.4$ for downstream estimates.*

4.2.8. Optimal Schur Weights and Exact Bounds

Theorem 4.18 (Exact values of C_1 and C_2) *With Schur weight $p_r = 1/|r|$:*

$$C_1 = \sup_{r\ge 1} G(r) = G(2) = \frac{5}{2}, \qquad C_2 = \sup_{s\ge 1} sF(s) = 5F(5) = \frac{6697}{1800}. \tag{214}$$

Hence

$$\|K\|_{\ell^2\to\ell^2} \le \sqrt{C_1C_2} = \frac{\sqrt{33485}}{60} \approx 3.0498. \tag{215}$$

Part 1 ($C_1 = 5/2$). From Lemma 4.9, $G(r) = 4H_r/r - 2/r^2$.
Direct values for $r \le 5$: $G(1) = 2$, $G(2) = 2.5$, $G(3) = 20/9 \approx 2.222$, $G(4) = 47/24 \approx 1.958$, $G(5) = 131/75 \approx 1.747$.
Monotonicity for $r \ge 5$. $G(r+1) - G(r) = f(r)/(r^2(r+1)^2)$, $f(r) = 4r^2 + 4r + 2 - 4H_r\cdot r(r+1)$. $f(5) = -152 < 0$. $f(r+1) - f(r) = 4r - 8(r+1)H_r < 0$ for $r \ge 5$ (since $8(r+1)H_r \ge 8\cdot 6\cdot H_5 \approx 109.6 > 4r = 20$). Hence $f(r) < 0$ for $r \ge 5$, so G decreasing on $[5,\infty)$. $\sup_{r\ge1} G(r) = G(2) = 5/2$. ✓

Part 2 (C_2). $sF(s) = 2H_s/s + 2H_s^{(2)} - 3/s^2$.
Direct values: $h(1) = 1$, $h(2) = 3.25$, $h(3) \approx 3.6111$, $h(4) \approx 3.7014$, $h(5) = 6697/1800 \approx 3.7206$, $h(6) \approx 3.7161$, $h(7) \approx 3.7032$, $h(8) \approx 3.6874$.
Monotonicity for $s \ge 5$. $h(s+1) - h(s) = -2H_s/(s(s+1)) + 1/(s+1)^2 + 3/s^2 < 0$ for $s \ge 5$ (verified via $-0.5667s^2 + 1.4333s + 3 < 0$ for $s \ge 4$). Hence $\sup_{s\ge1} h(s) = h(5) = 6697/1800$. ✓
Part 3 (norm). $\|K\| \le \sqrt{(5/2)(6697/1800)} = \sqrt{33485}/60 \approx 3.0498$. ✓

Theorem 4.19 (Improved boundedness of T) *For every $\alpha \in [0,2]$, $\|T\|_{Y_\alpha\to Y_\alpha} \le 3.05$.*

Stein interpolation with $M_0 = M_2 = 3.0498$; unitary equivalence.

Lemma 4.20 (Unitary equivalence) *$U_\alpha v_r |r|^{-\alpha/2} v_r$ unitary $\ell^2 \to Y_\alpha$; $K^{(\alpha)} = U_\alpha^{-1} T U_\alpha$.*

Direct computation.

4.2.9. Analytic Family

Definition 4.21 (Analytic family)
$K_{rs}^{(z)} |r|^{z/2-1}|s|^{-z/2}|r-s|^{-1}$ for $r \ne s$, $K_{rr}^{(z)} = 0$.

Lemma 4.22 (Analyticity and boundary bounds) *(a) Entire. (b) $\|K^{(i\tau)}\| = \|K^{(0)}\|$, $\|K^{(2+i\tau)}\| = \|K^{(2)}\|$. (c) $\|K^{(z)}\| \le \max(\|K^{(0)}\|, \|K^{(2)}\|)$.*

Direct computation.

4.3. Fixed-Point Formulation ($n \ge 1$)

4.3.1. Setup

$\sigma \in (0,1)$, $\mu > 1/2$, $\omega > 0$, $|\lambda| \le \Lambda_{\text{gap}}/2$, $|\gamma| \le \omega/175$. $e_n = A_n + \lambda A_n^\sigma$, $A_n = 4\omega n + \omega(2\mu+3)$.

4.3.2. Effective Energy and Composite Gap

Definition 4.23 (Effective energy and composite gap)
$g_k e_k + \gamma V_{kk}^{\log}$, $D_{n,r}(F) g_{n+r} - g_n - F$.

Lemma 4.24 (Composite gap at $\sigma = 1/2$) *For every $n \ge 0$, $r \ne 0$, $|F| \le R$:*

$$\boxed{|D_{n,r}(F)| \ge 2\omega|r| + \frac{\omega}{2}.} \tag{216}$$

At $|r| = 1$: $|D_{n,1}(F)| \ge 2.99\omega$.

Reverse triangle inequality: Terms 1 ($4\omega|r|$), 2 ($\le \omega|r|/2$), 3 ($\le 0.0054\omega|r|$), 4 ($\le \omega/2$). Sum: $3.4946\omega|r| - \omega/2 \ge 2\omega|r| + \omega/2$ for $|r| \ge 1$. At $|r| = 1$: $2.9946\omega \ge 2.99\omega$. ✓

Theorem 4.25 (Generalized composite gap) *Under (H1)–(H3) with $\sigma \in (0,1)$: $|D_{n,r}(F)| \ge 2\omega|r| + \omega/2$.*

Same decomposition; Term 2: $|\lambda|\sigma A_0^{\sigma-1}\cdot 4\omega|r| \le \omega|r|/2$.

4.3.3. Ansatz and Fixed-Point Equation

For $n \geq 1$, $|F| \leq R$, ansatz $\psi = \psi_n + \sum_{r\neq 0, n+r\geq 0} u_r \psi_{n+r}$, $E = g_n + F$. Projecting:

$$\boxed{(g_{n+r} - g_n - F)u_r = -\gamma V^{\log}_{n,n+r} - \gamma \sum_{s\neq r, n+s\geq 0} V^{\log}_{n+r,n+s} u_s.} \tag{217}$$

Definition 4.26 (Source and linear operator) $(A_n(F))_r - \gamma V^{\log}_{n,n+r}/D_{n,r}(F)$, $(B_n(F)u)_r - \gamma D_{n,r}(F)^{-1} \sum_{s\neq r, n+s\geq 0} V^{\log}_{n+r,n+s} u_s$. *Fixed-point:* $u = A_n(F) + B_n(F)u$.

4.3.4. Estimates

Lemma 4.27 (Estimate of $A_n(F)$) $|(A_n(F))_r| \leq |\gamma|/(2c_\omega r^2)$, $\|A_n(F)\|_{Z_2'} \leq 0.415|\gamma|/c_\omega$.

$|(A_n(F))_r| \leq |\gamma|(1/(2|r|))/(c_\omega |r|) = |\gamma|/(2c_\omega r^2)$. Norm: $1/(2(\log 3)^2) = 0.4143 \leq 0.415$.

Lemma 4.28 (Pointwise bound for $B_n(F)$) $|(B_n(F)u)_r| \leq |\gamma| \|u\|_{Z_2'}/(2c_\omega |r|) \sum_{s\neq r} (\log(2+|s|))^2/(|s|^2 |r-s|)$.

Lemma 4.29 (Convolution bound) *Sum* $\leq C_2^B(\log(2+|r|))^2/|r|$, $C_2^B \leq 22.4$.

Proposition 4.30 (Contraction of $B_n(F)$) $\|B_n(F)\|_{Z_2' \to Z_2'} \leq 11.2|\gamma|/c_\omega \leq 0.032$.

Combine and take sup.

4.3.5. Existence of $u_n(F)$

Theorem 4.31 (Existence and uniqueness) *For every $n \geq 1$, $F \in [-R, R]$, unique $u_n(F) \in Z_2'$ with $u_n(F) = A_n(F) + B_n(F)u_n(F)$.*

Banach.

Theorem 4.32 (Pointwise decay) $|u_{n,r}(F)| \leq C_{\text{dec}}(\log(2+|r|))^2/|r|^2$, $C_{\text{dec}} = 0.429|\gamma|/c_\omega$.

$\|u_n\| \leq 0.4143|\gamma|/(0.968c_\omega) = 0.4280|\gamma|/c_\omega \leq 0.429|\gamma|/c_\omega$.

Lemma 4.33 (Lipschitz of u_n) $\|u_n(F) - u_n(F')\|_{Z_2'} \leq L_u|F - F'|$, $L_u = 5.74 \times 10^{-4}/c_\omega$.

$\|u_n'\| \leq 1.0331(0.1854 + 0.00998)|\gamma|/c_\omega^2 = 0.2019|\gamma|/c_\omega^2$, $L_u = 0.2019/(350c_\omega) \leq 5.74 \times 10^{-4}/c_\omega$.

4.3.6. Consistency Function Φ_n

Definition 4.34 (Consistency function) $\Phi_n(F)\gamma \sum_{r\neq 0, n+r\geq 0} V^{\log}_{n,n+r} u_{n,r}(F)$.

Lemma 4.35 (Bound on Φ_n) $|\Phi_n(F)| \leq 0.772|\gamma|^2/c_\omega$.

Direct computation.

Lemma 4.36 (Lipschitz of Φ_n) $|\Phi_n(F) - \Phi_n(F')| \leq 2.95 \times 10^{-6}|F - F'|$.

$|\gamma| L_u \tilde{S}_2 = 2.95 \times 10^{-6}$.

Remark 4.37 (Uniformity) *Uniform in $n \geq 1$.*

4.3.7. Existence of F_n and Implicit Equation

Lemma 4.38 (Disk preservation) $|\Phi_n(F)| \leq 1.26 \times 10^{-5}\omega < R$.

Theorem 4.39 (Existence of F_n) *Unique $F_n = \Phi_n(F_n)$, $|F_n| \leq 1.26 \times 10^{-5}\omega$.*

Banach.

Remark 4.40 (Iteration) $F^{(k+1)} = \Phi_n(F^{(k)})$, *rate* 10^{-6}.

Remark 4.41 (Truncation error) $\kappa_\omega = 56C_2^B(\gamma_0^{\text{sharp}})^2/c_\omega = 56 \cdot 22.4/(2 \cdot 175^2)\omega \approx 2.05 \times 10^{-2}\omega$. *For* $\epsilon = 10^{-14}$, $R_{\max} \approx 3.4 \times 10^7$.

Theorem 4.42 (Exact implicit equation)

$$\boxed{E_n = g_n + F_n = e_n + \gamma V^{\log}_{nn} + F_n, \quad F_n = \Phi_n(F_n).} \tag{218}$$

Eigenvector $\psi \in \mathcal{D}(\mathcal{O}_0)$.

Five steps.

Corollary 4.43 (Uniqueness) E_n *unique in* $[g_n - R, g_n + R]$.

4.3.8. Analyticity

Theorem 4.44 (Analyticity of $E_n(\lambda, \gamma)$) *Analytic on* $P\{|\lambda| < \Lambda_{\text{gap}}/2, |\gamma| < \omega/175\}$.

Three stages. Stage 1: g_k affine. Stage 2: resolvent analytic via Neumann series. Stage 3: implicit function theorem (Zeidler, Berger).

Theorem 4.45 (Second-order expansion) $E_n = e_n + \gamma V^{\log}_{nn} + \gamma^2 E_n^{(2)} + O(\gamma^3)$ *with $E_n^{(2)}$ from Theorem 3.28.*

4.4. The Case $n = 0$

Index constraint $s \geq 1$.

4.4.1. Composite Gap at $n = 0$

Lemma 4.46 (Composite gap at $n = 0$) *For every $r \geq 1$, $|F| \leq R$: $|D_{0,r}(F)| \geq 2\omega r + \omega/2$.*

Same as Lemma 4.24.

4.4.2. Source and Contraction at $n = 0$

Lemma 4.47 (Improved source bound) $\|A_0(F)\|_{Z_2'} \leq 0.251|\gamma|/c_\omega$.

$|V^{\log}_{0,r}|^2 < 1/(4r^3)$; sum $\leq \zeta(3)/4 \approx 0.3005$; ratio $\sqrt{3\zeta(3)/\pi^2} \approx 0.6045$; $\|A_0\| \leq 0.6045 \cdot 0.415|\gamma|/c_\omega = 0.251|\gamma|/c_\omega$.

Proposition 4.48 (Contraction of $B_0(F)$) $\|B_0(F)\|_{Z_2' \to Z_2'} \leq 0.032$.

One-sided sum $\leq$ full sum.

4.4.3. Existence at $n = 0$

Theorem 4.49 (Existence of $u_0(F)$) *Unique $u_0(F) \in Z_2'$ with $|u_{0,r}(F)| \le C_{\mathrm{dec}}(\log(2+r))^2/r^2$.*

Banach.

Lemma 4.50 (Lipschitz of u_0) $L_u^{(0)} = 6.7 \times 10^{-4}/c_\omega$.

Same as Lemma 4.33.

Remark 4.51 (Relation to L_u) 6.7×10^{-4} *vs* 5.74×10^{-4}.

Theorem 4.52 (Existence of F_0) *Unique $F_0 = \Phi_0(F_0)$, $|F_0| \le 1.26 \times 10^{-5}\omega$.*

$\mathrm{Lip}(\Phi_0) \le 3.45 \times 10^{-6} < 1$.

Theorem 4.53 (Implicit equation at $n = 0$)

$$\boxed{E_0 = g_0 + F_0 = e_0 + \gamma V_{00}^{\log} + F_0, \quad F_0 = \Phi_0(F_0).} \tag{219}$$

Same as Theorem 4.42.

Remark 4.54 (Constants) *Exceptions:* $L_u^{(0)}$, $\mathrm{Lip}(\Phi_0) \le 3.45 \times 10^{-6}$.

Remark 4.55 (Geometric meaning) *One-sided sums; constants essentially unchanged.*

5. DISCUSSION AND CONCLUSION

In this final section we summarise the results, compare them with the existing literature, present potential applications, and list open questions.

5.1. Summary of Results

We have provided a complete spectral and perturbative analysis of the operator

$$\mathcal{H}(\lambda,\gamma) = \mathcal{O}_0 + \lambda\mathcal{O}_0^\sigma + \gamma \log x, \qquad 0 < \sigma < 1, \tag{220}$$

on the half-line $L^2(0,\infty)$, where $\mathcal{O}_0$ is the conformally invariant Schrödinger operator with singular Coulomb potential and harmonic confinement

$$\mathcal{O}_0 = -\frac{d^2}{dx^2} + \frac{\mu(\mu+1)}{x^2} + \omega^2 x^2, \qquad \mu > 1/2,\ \omega > 0. \tag{221}$$

Our results comprise the following eleven components, all with explicit constants.

(1) Explicit spectral analysis of $\mathcal{O}_0$. The base operator $\mathcal{O}_0$ is self-adjoint on $\mathcal{D}(\mathcal{O}_0)$, has purely discrete spectrum, and admits the orthonormal eigenbasis

$$\psi_n(x) = \mathcal{N}_n x^{\alpha+1/2} e^{-\omega x^2/2} L_n^{(\alpha)}(\omega x^2), \qquad A_n = 4\omega n + \omega(2\mu+3), \tag{222}$$

where $\alpha = \mu + 1/2$ and $\mathcal{N}_n = \sqrt{2\omega^{\alpha+1} n!/\Gamma(n+\alpha+1)}$. This is Theorem 2.13.

(2) Uniqueness of the self-adjoint extension. The limit-point nature of both endpoints (Lemma 2.25) forces

$$\mathcal{O}_0^{\min} = \mathcal{O}_0^{\max} = \mathcal{O}_0^F = \mathcal{O}_0, \tag{223}$$

including domains. This is Corollary 2.26 and Theorem 2.30.

(3) Relative boundedness with zero relative bound. Both perturbations satisfy, for every $\epsilon > 0$,

$$\|\mathcal{O}_0^\sigma f\| \le \epsilon\|\mathcal{O}_0 f\| + C_\epsilon\|f\|, \tag{224}$$

$$\|\log x \cdot f\| \le \epsilon\|\mathcal{O}_0 f\| + C_\epsilon'\|f\|, \tag{225}$$

for all $f \in \mathcal{D}(\mathcal{O}_0)$, with explicit constants $C_\epsilon^2 = \max(0, \sup_{t \ge A_0}(t^{2\sigma} - \epsilon^2 t^2))$ and $C_\epsilon' = \sqrt{L_{\delta,M}^2 + \epsilon^2/4}$. These are Theorems 3.1 and 3.3.

(4) Self-adjointness of the full operator. By the Kato–Rellich theorem, $\mathcal{H}(\lambda,\gamma)$ is self-adjoint on $\mathcal{D}(\mathcal{O}_0)$, bounded from below, with purely discrete spectrum. This is Proposition 3.5.

(5) Closed-form matrix elements. The logarithmic matrix elements $V_{mn}^{\log} = \langle \psi_m, \log x\, \psi_n\rangle$ admit the explicit expressions

$$V_{nn}^{\log} = \frac{1}{2}[\Psi(n+\alpha+1) - \log\omega], \tag{226}$$

$$V_{mn}^{\log} = -\frac{1}{2|m-n|}\sqrt{\frac{\max(m,n)!\,\Gamma(\min(m,n)+\alpha+1)}{\min(m,n)!\,\Gamma(\max(m,n)+\alpha+1)}}, \tag{227}$$

$$m \ne n, \tag{228}$$

where $\Psi = \Gamma'/\Gamma$ is the digamma function. This is Lemma 3.7.

(6) Uniform bound and regular expansion. The matrix elements satisfy the uniform bound

$$|V_{n,n+r}^{\log}| \le \frac{1}{2|r|}, \tag{229}$$

(Theorem 3.10) and the regular expansion

$$|V_{n,n+r}^{\log}|^2 = \frac{1}{4r^2}\left(1 + \frac{r}{n}\right)^{-\alpha}(1 + O(n^{-1})), \tag{230}$$

uniformly in $r \in [1,n]$ (Lemma 3.12). The matrix-element difference satisfies

$$\delta_r^{\mathrm{mat}} = |V_{n,n-r}^{\log}|^2 - |V_{n,n+r}^{\log}|^2 = -\frac{\alpha}{4n^2}(1 + O(\alpha r/n)), \tag{231}$$

exhibiting the asymmetry between $|V_{n,n-r}^{\log}|$ and $|V_{n,n+r}^{\log}|$ (Lemma 3.15).

(7) Complete second-order expansion. The n-th eigenvalue admits the expansion

$$E_n = e_n + \gamma V_{nn}^{\log} + \gamma^2 E_n^{(2)} + O(\gamma^3), \tag{232}$$

where

$$\boxed{E_n^{(2)} = -\frac{\lambda\pi^2\sigma(1-\sigma)}{24H_n^2}(4\omega n)^{\sigma-2} - \frac{\alpha(\log n + \gamma_E - \log 2)}{16\omega H_n^*(n+1)^2} + O(n^{2\sigma-3}) + O(n^{-2}).} \tag{233}$$

Here $H_n = 1 + \lambda\sigma(4\omega n)^{\sigma-1}$ and $H_n^* = 1 + \lambda\sigma A_n^{\sigma-1} = H_n + O(1/n)$. The algebraic term vanishes when $\lambda = 0$; the logarithmic term is present for all λ and is proportional to α. This is Theorem 3.28.

(8) Boundedness of the convolution operator. The convolution operator T introduced in Definition 4.6 satisfies

$$\|T\|_{Y_\alpha \to Y_\alpha} \leq 3.05 \tag{234}$$

for every $\alpha \in [0,2]$ (Theorem 4.19), and the Extended Convolution Lemma holds with $C_2^B \leq 22.4$ (Lemma 4.16).

(9) Fixed-point formulation. The eigenvalue problem for $\mathcal{H}(\lambda,\gamma)$ reduces to the contraction equation

$$u = A_n(F) + B_n(F)u \tag{235}$$

on the weighted Banach space Z_2', with $\|B_n(F)\|_{Z_2' \to Z_2'} \leq 0.032$ (Proposition 4.30). Consequently, the n-th eigenvalue satisfies the exact implicit equation

$$\boxed{E_n = g_n + F_n = e_n + \gamma V_{nn}^{\log} + F_n, \quad F_n = \Phi_n(F_n),} \tag{236}$$

with $\mathrm{Lip}(\Phi_n) \leq 2.95 \times 10^{-6}$ and $|F_n| \leq 1.26 \times 10^{-5}\omega$ (Theorem 4.39). The case $n = 0$ is treated separately with the same constants (Theorems 4.52– 4.53).

(10) Analyticity of the eigenvalues. The function $(\lambda,\gamma) \mapsto E_n(\lambda,\gamma)$ is analytic in a neighborhood of $(0,0)$ containing the polydisc

$$P = \left\{(\lambda,\gamma) \in \mathbb{R}^2 : |\lambda| < \frac{\Lambda_{\mathrm{gap}}}{2},\ |\gamma| < \frac{\omega}{175}\right\}. \tag{237}$$

This is Theorem 4.44.

(11) Generalization to $\sigma \in (0,1)$. The composite gap $|D_{n,r}(F)| \geq 2\omega|r| + \omega/2$ holds uniformly in $\sigma \in (0,1)$ (Theorem 4.25).

Summary of final constants.

- $C_V = 0.8607$ (digamma difference constant, Lemma 4.1);
- $C_1 = 5/2$, $C_2 = 6697/1800$, $\|K\| \leq \sqrt{33485}/60 \approx 3.0498$ (exact Schur constants, Theorem 4.18);
- $C_2^{\mathrm{tail}} = 2$ (sharp tail constant, Lemma 4.11);
- $C_2^B \leq 22.4$ (extended convolution constant, Lemma 4.16);
- $\|B_n(F)\| \leq 0.032$ (contraction constant, Proposition 4.30);
- $C_{\mathrm{dec}} \leq 0.429|\gamma|/c_\omega$ (decay constant, Theorem 4.32);
- $L_u = 5.74 \times 10^{-4}/c_\omega$ (Lipschitz constant, Lemma 4.33);
- $\|\Phi_n\|_\infty \leq 0.772|\gamma|^2/c_\omega$ (consistency bound, Lemma 4.35);
- $\mathrm{Lip}(\Phi_n) \leq 2.95 \times 10^{-6}$ (Lemma 4.36);
- $|F_n| \leq 1.26 \times 10^{-5}\omega$ (correction size, Theorem 4.39).

5.2. Comparison with the Literature

Our results complement and extend several classical bodies of work.

Kato's perturbation theory. The classical Kato–Rellich theorem [1] provides a general framework for relatively bounded perturbations with relative bound less than one. However, it does not provide explicit constants in specific applications, and it does not directly address the case of fractional powers or logarithmic operators. Our contribution fills this gap by providing the explicit constants C_ϵ and C_ϵ' (Theorems 3.1 and 3.3).

Reed–Simon spectral analysis. The standard treatment of confining Schrödinger operators [2–4] yields discreteness of the spectrum and completeness of eigenfunctions, but does not resolve the interaction of the eigenvalue problem with logarithmic perturbations, which is central to our analysis.

Sturm–Liouville theory. The reduction of $\mathcal{O}_0$ to Laguerre's equation is classical [21–23]. Our novel step is to exploit the explicit eigenbasis to obtain closed-form expressions for the logarithmic matrix elements in terms of the digamma function (Lemma 3.7) and to derive the complete second-order expansion (Theorem 3.28).

Schur test and Stein interpolation. The Schur test [29] is a classical tool for boundedness of integral operators. Our combination with Stein interpolation [30, 31] on a family of weighted kernels allows us to prove boundedness of the convolution operator on a scale of spaces Y_α uniformly in $\alpha \in [0,2]$ (Theorem 4.19). We further exploit the optimal Schur weight $p_r = 1/|r|$ to obtain the exact values of the row and column constants $C_1 = 5/2$ and $C_2 = 6697/1800$ (Theorem 4.18), which improve significantly over the crude bounds obtained with the standard weight $p_r = |r|^{-1/2}$.

Relation to [24]. The base operator with a single logarithmic perturbation was studied in [24] using WKB asymptotics with the Langer correction [25, 26]. That work established self-adjointness, purely discrete spectrum, a sharp asymptotic expansion, a modified Weyl law, and the associated Green's function. Our paper extends that work in three directions:

(i) We add a fractional perturbation $\lambda\mathcal{O}_0^\sigma$ with $0 < \sigma < 1$;

(ii) We replace the WKB-based analysis by a rigorous fixed-point formulation based on Kato–Rellich theory [1, 27, 28], the Schur test [29], Stein interpolation [30, 31], and the analytic implicit function theorem [57, 58];

(iii) We establish analyticity of $E_n(\lambda, \gamma)$ with explicit radius, derive the complete second-order expansion with both algebraic and logarithmic corrections, and generalize the composite gap to all $\sigma \in (0, 1)$.

The constants obtained here are substantially sharper: for example, the contraction constant is 0.032 (versus $1/4$ in earlier versions of the same program), and the extended convolution constant is $C_2^B \leq 22.4$ (versus 175).

Consistency with recent developments. Our results are consistent with recent advances in fractional operators [7, 38] and extend the analysis of fractional Schrödinger equations [8, 11, 12]. The modern literature on spectral asymptotics [39–49] confirms the relevance of the explicit-constant approach developed here. In particular, the sharp semiclassical spectral asymptotics of Mikkelsen [39] and the analysis of negative spectrum by Read [40] provide complementary perspectives on the eigenvalue distribution for Schrödinger operators with singular potentials.

Novelty. To the best of our knowledge, this is the first time the following results are obtained simultaneously for an operator combining fractional and logarithmic perturbations with a singular Coulomb potential:

(1) An explicit closed-form formula for the logarithmic matrix elements $V_{mn}^{\log}$ (Lemma 3.7);

(2) An exact fixed-point formulation of the eigenvalue problem with a uniform contraction constant 0.032 (Proposition 4.30);

(3) A complete second-order expansion with both algebraic and logarithmic corrections (Theorem 3.28);

(4) A rigorous proof of analyticity of $E_n(\lambda, \gamma)$ (Theorem 4.44);

(5) Explicit numerically usable constants for the eigenvalues and eigenvectors (Theorems 4.39 and 4.42);

(6) Generalization of the composite gap to all $\sigma \in (0, 1)$ (Theorem 4.25).

5.3. Potential Applications

(1) Numerical computation of eigenvalues. The implicit equation Eq. (236) provides an explicit iteration

$$F^{(k+1)} = \Phi_n(F^{(k)}), \qquad F^{(0)} = 0, \tag{238}$$

converging at rate $\mathrm{Lip}(\Phi_n) \leq 2.95 \times 10^{-6}$ (Lemma 4.36). Three iterations suffice for numerical accuracy of order $10^{-18}\omega$ (Remark 4.40). The correction size $|F_n| \leq 1.26 \times 10^{-5}\omega$ (Theorem 4.39) provides a rigorous a priori error bound for the eigenvalue approximation $E_n \approx e_n + \gamma V_{nn}^{\log}$.

(2) Quantum field theory on curved backgrounds. The operator $\mathcal{H}(\lambda, \gamma)$ arises as an effective generator in de Sitter-like geometries [20], where the fractional power $\mathcal{O}_0^\sigma$ and the logarithmic term $\gamma \log x$ model anomalous scaling corrections. The analyticity result (Theorem 4.44) guarantees perturbative expansion in the coupling constants, which is essential for renormalization group analysis.

(3) Stochastic analysis. Kolmogorov–Feller generators with logarithmic drift [18, 19] fit the general structure of $\mathcal{H}(\lambda, \gamma)$ with σ depending on the anomalous diffusion exponent. The explicit spectral gap $|D_{n,r}(F)| \geq 2\omega|r| + \omega/2$ (Theorem 4.25) translates into an explicit exponential convergence rate to equilibrium for the associated semigroup.

(4) Integrable systems. The modified Calogero–Sutherland model in the one-dimensional case [14–17] is closely related to $\mathcal{O}_0$, and our results extend the spectral analysis to a perturbed version of this model. The closed-form matrix elements (Lemma 3.7) are particularly useful in this context, as they allow explicit computation of correlation functions.

(5) Rigorous perturbation theory. The method developed here (closed-form matrix elements combined with a Schur/Stein interpolation argument) can be adapted to other unbounded perturbations whose kernel exhibits controlled decay. The key ingredients are:

- the existence of an explicit orthonormal eigenbasis;
- a slow (power-law) decay bound on the matrix elements;
- a summable weight that makes the associated convolution operator bounded.

The general framework is illustrated by the fixed-point formulation on the weighted space Z_2' (Definition 4.3).

5.4. Open Questions

We conclude with a list of open questions that naturally arise from the present work.

(1) The case $\mu \leq 1/2$. The domain condition $f/x^2 \in L^2$ becomes problematic at $\mu = 1/2$ and fails entirely at $\mu < 1/2$. It would be interesting to extend the analysis to the borderline case $\mu = 1/2$ (where uniqueness of the self-adjoint extension still holds) by relaxing the domain condition to $\mathcal{D}(\mathcal{O}_0^{\max})$, and to the limit-circle case $\mu < 1/2$ by selecting an appropriate boundary condition. In both cases, the eigenfunctions remain well-defined, but the domain of the operator requires careful analysis.

(2) Large logarithmic coupling. The contraction threshold $|\gamma| \leq \omega/175$ is restrictive. What happens when $|\gamma|$ exceeds this value? We expect the fixed-point operator to lose contraction, possibly leading to symmetry breaking or resonance phenomena. A natural conjecture is that the spectrum undergoes a phase transition at some critical value $\gamma_c > \omega/175$.

(3) Full fixed-point formulation for general $\sigma \in (0,1)$. While the composite gap (Theorem 4.25) and the second-order expansion (Theorem 3.28) are established for all $\sigma \in (0,1)$, the contraction estimates in §4.3 and §4.4 are derived under the specific assumption $\sigma = 1/2$. Extending the full fixed-point formulation (including the values of all constants) to general $\sigma \in (0,1)$ requires a more careful analysis of the σ-dependence of the quantities H_n, the composite gap, and the consistency function Φ_n.

(4) Sharp constants. Some of our constants (e.g., $C_2^B \leq 22.4$ versus the sharper 22.2 noted in Remark 4.17, or $C_{\text{emb}} \approx 583$ versus the true embedding constant) are conservative. Improving the calibration of these constants, particularly the threshold $\gamma_0^{\text{sharp}} = \omega/175$, is a natural next step. The optimal threshold is presumably determined by the spectral properties of the fixed-point operator $B_n(F)$ rather than by the crude Schur test estimates used here.

(5) Extension to higher-order expansions. Theorem 3.28 provides the second-order correction $E_n^{(2)}$. A natural extension is to compute the third-order correction $E_n^{(3)}$, which would require a more delicate analysis of the triple convolution $\Sigma_n^{(3)}$ defined in Eq. (183). The methods developed here (partial-fraction decomposition, discrete Hilbert transform) suggest that the third-order correction admits a similar closed-form structure, with contributions from both the algebraic and logarithmic terms.

(6) Analyticity radius. Theorem 4.44 shows that the polydisc P is contained in the domain of analyticity. Is P maximal, or can the radius be extended? The constraint $|\gamma| < \omega/175$ arises from the contraction estimate, not from a genuine singularity of the eigenvalue function. It is plausible that the true radius of analyticity is determined by the first eigenvalue collision or the first singularity of the resolvent.

5.5. Conclusion

We have established a complete spectral and perturbative analysis of the operator $\mathcal{H}(\lambda, \gamma)$ on the half-line, combining fractional and logarithmic perturbations with a singular Coulomb potential and a harmonic confinement. Our approach yields explicit, numerically usable constants and provides a constructive fixed-point formulation of the eigenvalue problem. The results open several directions for further investigation, including the borderline case $\mu = 1/2$, the large logarithmic coupling regime, and the general $\sigma \in (0,1)$ case for the full fixed-point formulation.
The methodology developed in this paper – combining the explicit Laguerre eigenbasis, closed-form matrix elements in terms of the digamma function, Schur/Stein interpolation for the convolution operator, and the analytic implicit function theorem – provides a template for rigorous perturbation theory of singular Schrödinger operators with unbounded perturbations. We hope that these techniques will find applications in related problems, such as random Schrödinger operators with logarithmic correlations, or the spectral theory of fractional powers of confining operators on more general domains.

A. CONSTANTS AND TABLES

A.1. Table of Main Constants

A.2. Numerical Constants

A.3. Dependence of the Constants on ω

The dependence of the constants on ω (with $|\gamma| = \omega/175$) is summarized in Table 3 in §2.1.5. The key points are:

- Constants derived from the fixed-point structure ($\|B_n\|$, $\text{Lip}(\Phi_n)$) are independent of ω thanks to the calibration $\gamma_0^{\text{sharp}} = \omega/175$;
- Constants derived from the gap ($\|A_n\|$, C_{dec}, L_u, $\|\Phi_n\|_\infty$) scale inversely with ω;
- Constants C_V, C_1, C_2, $\|K\|$, C_2^B, C_2^{tail}, $\tilde{S}_2$, S_2, C_{emb} are independent of ω (dimensionless or depending only on the structure of the operator).

A.4. Verification of Numerical Bounds

We provide numerical verifications of the standard constants used in the estimates.

Verification of $\log 3 < 1.0987$. The partial sum of the exponential series $\sum_{k=0}^{12} u^k/k!$ at $u = 1.0987$ exceeds $3.00002 > 3$, hence $\log 3 < 1.0987$.

Verification of γ_E. Using the Euler–Maclaurin formula at $n = 10$: $H_{10} = 7381/2520 \approx 2.9289682$. Hence $\gamma_E \in (0.5764, 0.5773)$.

Verification of $\zeta(3/2)$. Using partial sums up to $N = 10$: $S_{10} \approx 1.9953$. With integral tail: $\zeta(3/2) \approx 2.6124$.

Verification of $\zeta(3)$. Using partial sums up to $N = 10$: $S_{10} \approx 1.1975$. With Euler–Maclaurin tail: $\zeta(3) \approx 1.2021$.

Verification of $\zeta(7/4)$. Using partial sums up to $N = 10$: $S_{10} \approx 1.7338$. With Euler–Maclaurin tail: $\zeta(7/4) \approx 1.9680$.

A.5. Supplementary Derivations

A.5.1. Derivation of Λ^*
See the end of §2.1.4.

A.5.2. Derivation of C_V
See the proof of Lemma 4.1.

A.5.3. Derivation of C_2^B
See the proof of Lemma 4.16.

Table[4]: All explicit constants used in the paper. The classification follows the conventions stated at the beginning of Section 2: exact/sharp values, rigorous bounds, and working bounds derived from the fixed-point structure.

Constant	Value / Bound	Location
C_V (digamma difference)	0.8607 (sharp)	Lemma 4.1
C_1 (Schur row)	$5/2 = 2.5$ (exact)	Theorem 4.18
C_2 (Schur column)	$6697/1800 \approx 3.7206$ (exact)	Theorem 4.18
$\|K\|$ (Schur norm)	$\leq \sqrt{33485}/60 \approx 3.0498$	Theorem 4.18
$\|T\|_{Y_\alpha \to Y_\alpha}$	≤ 3.05	Theorem 4.19
C_2^{tail} (tail constant)	≤ 2	Lemma 4.11
C_2^B (extended convolution)	≤ 22.4 (sharp ≈ 22.2)	Lemma 4.16
$\tilde{S}_2$ (logarithmic sum, s^{-3})	< 1.80	Lemma 4.13
S_2 (logarithmic sum, s^{-2})	< 8.92	Lemma 4.14
C_{emb} (embedding)	≈ 583	Lemma 4.4
$\|B_n(F)\|$ (contraction, $n \geq 1$)	≤ 0.032	Proposition 4.30
$\|B_0(F)\|$ (contraction, $n = 0$)	≤ 0.032	Proposition 4.48
C_{dec} (decay)	$\leq 0.429\|\gamma\|/c_\omega$	Theorem 4.32
L_u (Lipschitz, $n \geq 1$)	$\leq 5.74 \times 10^{-4}/c_\omega$	Lemma 4.33
$L_u^{(0)}$ (Lipschitz, $n = 0$)	$\leq 6.7 \times 10^{-4}/c_\omega$	Lemma 4.50
$\|\Phi_n\|_\infty$ (consistency)	$\leq 0.772\|\gamma\|^2/c_\omega$	Lemma 4.35
$\mathrm{Lip}(\Phi_n)$ $(n \geq 1)$	$\leq 2.95 \times 10^{-6}$	Lemma 4.36
$\mathrm{Lip}(\Phi_0)$ $(n = 0)$	$\leq 3.45 \times 10^{-6}$	Theorem 4.52
$\|F_n\|$ (correction size)	$\leq 1.26 \times 10^{-5}\omega$	Theorem 4.39
$\|A_n(F)\|$ (source, $n \geq 1$)	$\leq 0.415\|\gamma\|/c_\omega$	Lemma 4.27
$\|A_0(F)\|$ (source, $n = 0$)	$\leq 0.251\|\gamma\|/c_\omega$	Lemma 4.47
κ_ω (truncation)	$\approx 2.05 \times 10^{-2}\omega$	Remark 4.41

Table[5]: Accurate numerical values of standard constants used in the estimates throughout the paper.

Constant	Value
$\log 2$	0.6931
$\log 3$	1.0986
$\log 4$	1.3863
$\log 5$	1.6094
$\log 6$	1.7918
$\log 7$	1.9459
$\log 8$	2.0794
$\zeta(3/2)$	2.6124
$\zeta(3)$	1.2021
$\zeta(7/4)$	1.9680
γ_E (Euler–Mascheroni)	0.5772

A.5.4. Derivation of C_2^{tail}

See the proof of Lemma 4.11.

A.5.5. Derivation of κ_ω

The truncation constant κ_ω from Remark 4.41 is

$$\kappa_\omega \frac{56\, C_2^B\, (\gamma_0^{\mathrm{sharp}})^2}{c_\omega} = \frac{56 \cdot 22.4}{2 \cdot 175^2}\,\omega \approx 2.05 \times 10^{-2}\,\omega. \tag{239}$$

The factor $\gamma_0^{\mathrm{sharp}} = \omega/175$ has units of energy, so κ_ω also has units of energy. For $\epsilon = 10^{-14}$ and $\omega = 1$: $R_{\max} \approx 3.4 \times 10^7$.

A.5.6. Numerical value of C_2

We verify numerically that $C_2 = 5F(5) = 6697/1800$. Using the closed form Eq. (205):

$$F(5) = \frac{2H_5}{25} + \frac{2H_5^{(2)}}{5} - \frac{3}{125}.$$

With $H_5 = 137/60$ and $H_5^{(2)} = 5269/3600$:

$$F(5) = \frac{2 \cdot 137}{25 \cdot 60} + \frac{2 \cdot 5269}{5 \cdot 3600} - \frac{3}{125}.$$

Simplifying with common denominator 18000:

$$F(5) = \frac{3288 + 10538 - 432}{18000} = \frac{13394}{18000} = \frac{6697}{9000}.$$

Hence $5F(5) = 6697/1800 \approx 3.72056$. ✓

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